

Robust utility maximization without model compactness

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Abstract

We formulate conditions for the solvability of the problem of robust utility maximization from final wealth in continuous time financial markets, without assuming weak compactness of the densities of the uncertainty set, as customary in the literature. Relevant examples of such a situation typically arise when the uncertainty set is determined through moment constraints. Our approach is based on identifying functional spaces naturally associated with the elements of each problem. For general markets these are modular spaces, through which we can prove a minimax equality and the existence of optimal strategies by exploiting the compactness, which we establish, of the image by the utility function of the set of attainable wealths. In complete markets we obtain additionally the existence of a worst-case measure, and combining our ideas with abstract entropy minimization techniques, we moreover provide in that case a novel methodology for the characterization of such measures.

Keywords: Robust utility maximization, non-compact uncertainty set, modular space, Orlicz space, worst-case measure, entropy minimization

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1 Introduction

The problem of utility maximization in continuous time models of financial markets has been thoroughly researched in the last decades. However, in a standard utility maximization problem one is forced to choose (or say fix) a probability measure under which the random objects in the model shall evolve. In practical terms it is next to impossible to, with complete accuracy, compute the real-world measure. For instance any statistical method shall only sign out a region of confidence for it. Therefore one is quickly led to consider utility maximization under families of possible measures (we refer to this as the *uncertainty set* or *set of priors*, usually denoted \mathcal{Q}) rather than over a unique a priori one; see [19] for more on this idea. A commonly adopted (though very conservative) point of view is to look for strategies that are optimal in the worst possible sense:

$$\text{maximize } \inf_{Q \in \mathcal{Q}} \mathbb{E}^Q[\text{utility}(X)] \text{ over all admissible terminal wealths } X \text{ starting at } x.$$

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We will also consider here such a point of view and, as usual in the literature, we shall refer to this stochastic optimization problem as the robust variant of the (standard, non-robust) utility maximization one.

In [17, 21, 36, 38, 39], to name a few, the problem of robust utility maximization from terminal wealth is solved in a way that greatly recovers the results known for the non-robust situation. The authors successfully apply convex-duality arguments and deliver attainability of the problem (as well as of its dual, conjugate problem) and even the existence of what may be called a “worst-case measure”; this is, a measure in the given family for which the optimal utility is as low as it gets. In presence of consumption, the problem has also been considered in e.g. [11, 40]. Robust portfolio optimization problems have also been studied by using other tools, see e.g. [22] for a stochastic control approach (via PDEs), as well as [10] and the references therein for an approach using BSDEs. The case when the uncertainty set is not dominated by a single reference measure, motivated by the issue of misspecification of volatilities, was popularized by [15], where it was studied under a tightness hypothesis.

Whatever the approach, some type of compactness assumption on the family of possible measures seems prevalent in most of the aforementioned works, the usual assumption in the dominated case being that the densities of the laws in the uncertainty set form a uniformly integrable set. However, even extremely simple instances of the problem suggests that this assumption is too stringent (see Example 2.7). Moreover, very little concrete information is known about the worst-case measure, beyond very specific instances of the problem, despite the fact the dual of the robust utility maximization problem that it solves actually is a “convex problem” (namely to minimize a convex functional under linear-convex constraints) when seen as an infinite-dimensional optimization problem.

In the present work, we will restrict ourselves to the dominated case and we will only consider utilities on the positive half-line. In this setting, we will introduce a unified functional framework for the robust portfolio optimization problem that naturally copes with part of the aforementioned non-satisfactory aspects of the available literature. Our approach will be based on finding an appropriate Banach space where hypothetical worst-case measures should a fortiori lie. This space will turn out to be a convex modular space (see [31]), and it will be closely related to the optimization problems at hand, more concretely, to the convex dual problem related to the Legendre transform of the utility function. In this setting, the robust utility maximization problem will reduce to solving:

$$\text{maximize } \inf_{Q \in \mathcal{Q}} \mathbb{E}^{\mathbb{P}} \left[\frac{dQ}{d\mathbb{P}} K \right] \text{ over } K \in \mathcal{K}$$

where \mathcal{K} is the image through the utility function of all possible terminal wealths (with common initial starting point). The crucial argument, as well as the point where most mathematical difficulties arise, is to provide verifiable conditions on the utility function and the market under which \mathcal{K} is a weakly compact set in the norm-dual of the mentioned modular space. We will rely on this approach in Theorem 2.4 to prove the usual minimax equality as well as the existence of optimal wealth processes and conjugacy of value functions. We thus extend some of the results in [17, 39] roughly assuming that the densities of the uncertainty set be contained in the modular space and that they form a weakly closed set with respect to this topology, instead of the usual compactness assumption (we thank a referee for pointing out that the argument in [39] for the existence of optimal wealths actually holds without compactness as well), and we do so without relying on the existence of a saddle point (the worst case measure) or on any assumption implying this. We envision that this functional point of view and the described compactness of \mathcal{K} should thus open the way to new applications. Indeed, already the characterization of

worst-case measures in the complete case, which we will carry out in the present work, is only possible thanks to the functional setting we adopt, and the compactness of \mathcal{K} has been crucially applied in [5] in the context of sensitivity analysis.

When aiming to recover those results in [17, 39] not covered by our Theorem 2.4, for instance the existence of a worst-case measure, we realize that replacing the usual compactness assumption by reflexivity of the modular space is a sufficient condition to do this. In this respect we prove, modulo some pathologies on the filtered probability space, that our modular spaces are unfortunately never reflexive for strict incomplete markets; this is the content of Theorem 2.5 (more specifically Theorem 5.14 and the remarks thereafter).

On the positive side, when we specialize our analysis to complete markets, our modular spaces become Orlicz-Musielak spaces and we can provide easily verifiable conditions under which they become reflexive. Of course, Orlicz spaces are well known about in Mathematical Finance (see e.g. [12] regarding risk measures, [20] on utility maximization and [8] on admissibility of trading strategies). Related to our work, in [17, 21] Orlicz spaces arise in connection to the Vallee Poussin criterion when studying the problem by means of f -divergences, and we will comment more about this in Section 2.2. Our choice of an Orlicz space, in the complete case, obeys different considerations and makes a more systematic use of the properties of the space in connection to the robust problem; furthermore, our functional setting will be crucial for the new application which we have already hinted at and which we discuss next.

Using our Orlicz space formulation of the dual (minimization) problem for complete markets we will give, in the reflexive setting, a novel and explicit characterization of the worst-case measure that covers a much broader range of applications than is available in the literature. More precisely, by writing the general set of possible models \mathcal{Q} in terms of a potentially infinite system of linear constraints (that may be thought of as moment constraints on some market observables or insider information), we will be able to adapt to the financial framework some general entropy minimization techniques developed in [28, 29] and characterize in Theorem 2.10 the worst-case measure $\hat{\mathbb{Q}} \in \mathcal{Q}$ in terms of a related abstract concave maximization problem. We may call it with some abuse the *dual of a dual* problem. By finding a solution g to that problem we obtain the expression:

$$\hat{\mathbb{Q}} = \text{risk-neutral density} \times [U^{-1}]'(\text{linear operator}(g)),$$

where the *linear operator* above describes how the element g acts upon the observables of the market that we use to describe (through their moments) the set \mathcal{Q} . The so-called dual of a dual problem may in many practical situations be easier to solve than the original one; for instance, it is finite-dimensional if \mathcal{Q} is specified by finitely many constraints.

The remainder of the paper is organized as follows. We start Section 2 describing the mathematical framework of the robust optimization problem in continuous-time financial markets following [39] and introduce the basic notation required throughout. Then in Section 2.1 we will state our main results about incomplete markets, in 2.2 we compare them with the existing literature and finally in 2.3 state our specialized results for complete markets. In Section 3.1 we recall some known properties of Orlicz-Musielak spaces and in 3.2 provide results of our own connecting them to our robust problem. Our main results on the robust optimization problem in the complete case are then established and proven in Section 4. In Section 5 we introduce the modular spaces associated with the incomplete case and study them extensively. Both Sections 4 and 5 are independent of each other, and the reader can skip either of them depending on which result he/she is interested in. Finally, some technical facts are proved in the Appendix.

2 Preliminaries and statement of main results

We will work in a similar setting as [25, 39]. Let there be d stocks and a bond, normalized to one for simplicity. Let $S = (S^i)_{1 \leq i \leq d}$ be the price process of these stocks, and $T < \infty$ a finite investment horizon. The process S is assumed to be a semimartingale in a filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$, where \mathbb{P} will always stand for the *reference measure*. The expectation with respect to \mathbb{P} will be denoted by \mathbb{E} . The set of all probability measures on (Ω, \mathbb{F}) absolutely continuous w.r.t \mathbb{P} will be denoted by \mathcal{P} , and the expectation with respect to $\mathbb{Q} \in \mathcal{P} \setminus \{\mathbb{P}\}$ will be expressed by $\mathbb{E}^{\mathbb{Q}}$.

A (self-financing) portfolio π is defined as a couple (X_0, H) , where $X_0 \geq 0$ denotes the (constant) initial value associated to it and $H = (H^i)_{i=1}^d$ is a predictable and S -integrable process which represents the number of shares of each type under possession. The wealth associated to a portfolio π is the process $X = (X_t)_{t \leq T}$ given by

$$X_t = X_0 + \int_0^t H_u dS_u \quad (2.1)$$

and the set of attainable wealths from x is defined as

$$\mathcal{X}(x) = \{X \geq 0 : X \text{ as in (2.1) s.t. } X_0 \leq x\}. \quad (2.2)$$

The set of equivalent local martingale measures (or risk neutral measures) associated to S is

$$\mathcal{M}^e(S) = \{\mathbb{P}^* \sim \mathbb{P} : \text{every } X \in \mathcal{X}(1) \text{ is a } \mathbb{P}^*\text{-local martingale}\} \quad (2.3)$$

which reduces to

$$\mathcal{M}^e(S) = \{\mathbb{P}^* \sim \mathbb{P} : S \text{ is a } \mathbb{P}^*\text{-local martingale}\}$$

if S is locally bounded. This is assumed in all the sequel, together with the fact that the market is *arbitrage-free* in the sense of NFLVR, meaning that $\mathcal{M}^e(S)$ is not empty.

As usual the market model is coined *complete* if $\mathcal{M}^e(S)$ is reduced to a singleton, i.e. $\mathcal{M}^e(S) = \{\mathbb{P}^*\}$. Given $\mathbb{Q} \in \mathcal{P}$, the following set generalizes the set of density processes (with respect to \mathbb{Q}) of risk neutral measures equivalent to it:

$$\mathcal{Y}_{\mathbb{Q}}(y) := \{Y \geq 0 | Y_0 = y, XY \text{ is } \mathbb{Q} - \text{supermartingale } \forall X \in \mathcal{X}(1)\}.$$

Introduced in [25], $\mathcal{Y}_{\mathbb{Q}}(y)$ plays a central role in portfolio optimization in incomplete markets.

Definition 2.1. A function $U : (0, \infty) \rightarrow \mathbb{R}$ is called a *utility function on $(0, +\infty)$* , if it is strictly increasing, strictly concave and continuously differentiable. It will be said to satisfy *INADA* if

$$U'(0+) = \infty \text{ and } U'(+\infty) = 0.$$

Such a function U is always extended as $-\infty$ on $(-\infty, 0)$. Its *asymptotic elasticity*, introduced in [25], is defined as $AE(U) := \limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)}$. Last, if $\Delta := \lim_{x \rightarrow +\infty} U(x) < \infty$, we set $U^{-1} = +\infty$ on $[\Delta, \infty)$.

Suppose now that an agent aims to optimize the utility U of her final wealth, by investing during a time interval $[0, T]$ in a market which might be described by more than one probabilistic model (the actual or more accurate one being unknown to her). Let $\mathcal{Q} \subset \mathcal{P}$ be a set of feasible probability measures on $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t \leq T}, \mathbb{P})$ representing the

mentioned ambiguity or uncertainty. We shall refer to such a set as the *uncertainty set* from here on. A common paradigm is that the agent tries to maximize the worst-case expected utility given the set of models under consideration, by solving the optimization problem

$$\sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}(U(X_T)), \quad (2.4)$$

(a suitable meaning can be given to the expectation in case U is unbounded). Throughout the present work it will be assumed that \mathcal{Q} contains only probability measures that are absolutely continuous with respect to \mathbb{P} . We will write

$$\mathcal{Q}_e := \{\mathbb{Q} \in \mathcal{Q} | \mathbb{Q} \sim \mathbb{P}\}$$

and respectively denote by

$$\frac{d\mathcal{Q}}{d\mathbb{P}} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q} \right\}, \quad \frac{d\mathcal{Q}_e}{d\mathbb{P}} := \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \mathbb{Q} \in \mathcal{Q}_e \right\} = \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} \in \frac{d\mathcal{Q}}{d\mathbb{P}} : \frac{d\mathbb{Q}}{d\mathbb{P}} > 0 \text{ a.s.} \right\}.$$

the set of densities with respect to \mathbb{P} of the elements of \mathcal{Q} and \mathcal{Q}_e . As in the standard, non-robust, setting (see [35] for general background), the dual formulation of the optimization problem (2.4) will make use of the conjugate function of U , given by

$$V(y) := \sup_{x > 0} [U(x) - xy] \quad \forall y > 0$$

(actually the Fenchel conjugate of $-U(-\cdot)$). The following functions commonly used in the literature to tackle problem (2.4), will also be relevant here:

$$\begin{aligned} u(x) &= \sup_{X \in \mathcal{X}(x)} \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}(U(X_T)) & u_{\mathbb{Q}}(x) &= \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}(U(X_T)), \\ v_{\mathbb{Q}}(y) &= \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}(V(Y_T)) & v(y) &= \inf_{\mathbb{Q} \in \mathcal{Q}_e} v_{\mathbb{Q}}(y). \end{aligned} \quad (2.5)$$

Of course, $u_{\mathbb{Q}}(x)$ is the investor's *subjective* utility under model $\mathbb{Q} \in \mathcal{Q}_e$, when starting from an initial wealth not larger than $x > 0$, whereas $u(x)$ is her robust utility. The function $x \mapsto u_{\mathbb{Q}}(x)$ is concave (as an easy check shows), so that $u_{\mathbb{Q}}(x_0) < +\infty$ at some $x_0 > 0$ for some given $\mathbb{Q} \in \mathcal{Q}$ implies $u_{\mathbb{Q}} < +\infty$ and then, $u < +\infty$, by the usual min-max inequality.

For a fixed $\mathbb{Q} \in \mathcal{Q}_e$ it was proven in Theorem 3.1 of [25] that $u_{\mathbb{Q}}$ and $v_{\mathbb{Q}}$ are conjugate:

$$u_{\mathbb{Q}}(x) = \inf_{y > 0} (v_{\mathbb{Q}}(y) + xy) \quad \text{and} \quad v_{\mathbb{Q}}(y) = \sup_{x > 0} (u_{\mathbb{Q}}(x) - xy) \quad (2.6)$$

whenever $u_{\mathbb{Q}}$ is finite. Hence, since the inequalities

$$\begin{aligned} u(x) &\leq \inf_{y > 0} \left(\inf_{\mathbb{Q} \in \mathcal{Q}} \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}(V(Y_T)) + xy \right) \\ &\leq \inf_{y > 0} \left(\inf_{\mathbb{Q} \in \mathcal{Q}_e} \inf_{Y \in \mathcal{Y}_{\mathbb{Q}}(y)} \mathbb{E}^{\mathbb{Q}}(V(Y_T)) + xy \right) = \inf_{y > 0} (v(y) + xy) \end{aligned} \quad (2.7)$$

always hold, the function v can be considered as a candidate conjugate of u .

We will denote in the sequel by $L^0 = L^0(\Omega, \mathbb{P})$ the space of measurable functions equipped with the topology of convergence in probability, and by $L_+^0 \subset L^0$ the cone of non-negative functions therein. We shall also write

$$\mathcal{Y} := \mathcal{Y}_{\mathbb{P}}(1),$$

and we will often use Y instead of Y_T , which should be clear from context. In order to state the assumptions that will hold throughout this work we will also need the subset

$$\mathcal{Y}^* := \{Y \in \mathcal{Y} : Y > 0 \text{ a.s. and } \forall \beta > 0, \mathbb{E}[V(\beta Y)] < \infty\}. \quad (2.8)$$

2.1 Main results in general markets

We start noting that for every $\mathbb{Q} \in \mathcal{Q}_e$, we have $\mathcal{Y}_{\mathbb{Q}}(y) = \{ \frac{yY}{Z^{\mathbb{Q}}} : Y \in \mathcal{Y}_{\mathbb{P}}(1) \}$, where $Z^{\mathbb{Q}}$ is the density process of \mathbb{Q} w.r.t. \mathbb{P} , hence

$$v(y) = \inf_{\mathbb{Q} \in \mathcal{Q}_e} \inf_{Y \in \mathcal{Y}_{\mathbb{P}}(1)} \mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} V \left(y Y_T \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right]^{-1} \right) \right]. \quad (2.9)$$

Thus, if v is to be finite at some point $y > 0$, the only measures \mathbb{Q} that matter in (2.9) are those such that, for some $Y \in \mathcal{Y}_{\mathbb{P}}(1)$,

$$\mathbb{E}^{\mathbb{P}} \left[\frac{d\mathbb{Q}}{d\mathbb{P}} V \left(y Y_T \left[\frac{d\mathbb{Q}}{d\mathbb{P}} \right]^{-1} \right) \right] < \infty.$$

This motivates us to restrict from the outset the set \mathcal{Q} to consist of measures \mathbb{Q} for which $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is in the space of measurable functions

$$L_I := \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \inf_{Y \in \mathcal{Y}} \mathbb{E}[|Z|V(Y/(\alpha|Z|))] < \infty \right\} = \bigcup_{Y \in \mathcal{Y}} L_{|\cdot|V \circ Y/|\cdot|},$$

where for every $Y \in \mathcal{Y}$ we define:

$$L_{|\cdot|V \circ Y/|\cdot|} := \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^{\mathbb{P}}[|Z|V(Y/(\alpha|Z|))] < \infty \right\}.$$

We will see in Section 3 that the function $z \mapsto |z|V(Y/|z|)$ is a.s. non-negative and convex and that $L_{|\cdot|V \circ Y/|\cdot|}$ turns out to be an Orlicz-Musielak space (see Remark 3.12) for each $Y \in \mathcal{Y}^*$. In particular, it is a Banach space with the adequate norms; properties of these spaces (which can be seen as Orlicz spaces based on “random Young functionals”) will be recalled in Theorem 3.4. The convex conjugate of $|\cdot|V \circ Y/|\cdot|$ will be shown in Lemma 3.9 to be the function $YU^{-1} \circ |\cdot|$, and it will play a pre-eminent role, as will do the associated Orlicz-Musielak space

$$L_{YU^{-1} \circ |\cdot|} := \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}[YU^{-1}(\alpha|Z|)] < \infty \right\}.$$

The following assumption will be relevant in the study of topological duality between the spaces $L_{|\cdot|V \circ Y/|\cdot|}$ and $L_{YU^{-1} \circ |\cdot|}$. It is not assumed to hold, unless specifically stated:

Assumption 2.2. For some constants $a, b, k, d > 0$, the convex functions $V(\cdot)$ and $U^{-1}(\cdot)$ on $(0, \infty)$ satisfy for all $y > 0$:

$$V(y/2) \leq aV(y) + b(y+1) \quad (2.10)$$

$$U^{-1}(2y) \leq kU^{-1}(y) + d. \quad (2.11)$$

In the jargon of Orlicz space theory (see e.g. [37]), Assumption 2.2 correspond to “ Δ_2 and ∇_2 ” conditions on the Young function $|\cdot|V \circ 1/|\cdot|$. As pointed out in Theorem 3.6, Assumption 2.2 implies reflexivity of $L_{|\cdot|V \circ Y/|\cdot|}$ and $L_{YU^{-1} \circ |\cdot|}$, and is necessary for the latter if \mathbb{P} is atomless.

The space L_I will be endowed with a suitable Banach space topology called Modular Space topology which generalizes the Orlicz-Musielak one (see Section 5.1) and tightly harmonizes with our optimization problem. The search for verifiable conditions on the function U that may render the space L_I to be tractable will lead us to introduce the space

$$L_J := \left\{ Z \in L^0 \text{ s.t. } \exists \alpha > 0, \sup_{Y \in \mathcal{Y}} \mathbb{E}[YU^{-1}(\alpha|Z|)] < \infty \right\} \subseteq \bigcap_{Y \in \mathcal{Y}} L_{YU^{-1} \circ |\cdot|}.$$

In general L_J is included in the algebraic dual of L_I . Under Assumption 2.3 below L_J is actually included in the topological dual of L_I (cf. Proposition 5.8), and if additionally (2.10) holds, the latter space and L_J will be precisely isometric isomorphic (cf. Proposition 5.9). Denoting by $\sigma(L_I, L_J)$ the weak topology on L_I induced by L_J , we now state our main hypothesis:

- Assumption 2.3.** 1. U is a utility function on $(0, \infty)$ satisfying INADA and such that $U(0+) = 0$.
2. The set \mathcal{Y}^* is a non-empty subset of $\mathcal{Y}_{\mathbb{P}}(1)$.
3. Regarding \mathcal{Q} we assume:
- (a) \mathcal{Q} is countably convex.
 - (b) $[\mathbb{P}(A) = 0 \iff \forall \mathbb{Q} \in \mathcal{Q}, \mathbb{Q}(A) = 0]$.
 - (c) $d\mathcal{Q}/d\mathbb{P}$ is a non-empty $\sigma(L_I, L_J)$ -closed subset of L_I
 - (d) $\exists x > 0, \mathbb{Q} \in \mathcal{Q}_e$ such that $u_{\mathbb{Q}}(x) < \infty$

Exploiting a certain compactness of the image under U of the terminal wealths, as elements in L_J , our main result for general markets, proved in Section 5.3, will establish the minimax equality and the existence of optimal strategies:

Theorem 2.4. *Suppose Assumption 2.3 holds. Assume moreover that $L_I^* \cong L_J$, which is true as soon as (2.10) in Assumption 2.2 additionally holds and in particular if $AE(U) < 1$. Then for every $x > 0$:*

$$\begin{aligned} u(x) &= \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}(U(X_T)) = \inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}\left(U\left(\hat{X}_T\right)\right) \\ &= \inf_{\mathbb{Q} \in \mathcal{Q}_e} \sup_{X \in \mathcal{X}(x)} \mathbb{E}^{\mathbb{Q}}(U(X_T)) < +\infty, \end{aligned} \quad (2.12)$$

for some $\hat{X} \in \mathcal{X}(x)$. Moreover v is finite and u, v are conjugate on $(0, \infty)$. Furthermore if L_I is reflexive, which happens as soon as the market is complete and the full Assumption 2.2 holds, then there is a saddle point, i.e. there exists a unique $\hat{\mathbb{Q}} \in \mathcal{Q}$ so that all the values in (2.12) equal to $\mathbb{E}^{\hat{\mathbb{Q}}}\left[U\left(\hat{X}_T\right)\right]$.

Our second main result in the setting of incomplete markets, however, is of a negative kind. It states that reflexivity of L_I is virtually impossible in most strict incomplete market models, independently of how good the utility functions is. This is quite remarkable since it implies that the route, through reflexivity, to establish the existence of a saddle point when the set \mathcal{Q} is only weakly closed in L_I (as by the end of the previous theorem), is feasible if and only if the market is complete to begin with:

Theorem 2.5. *Under parts 1. and 2. of Assumption 2.3, if the set \mathcal{Y} is not uniformly integrable, then L_I cannot be reflexive.*

As it shall be discussed in Section 5, in most reasonable strictly incomplete market models (for instance those involving the brownian filtration) \mathcal{Y} is indeed never uniformly integrable.

In the complete case, in turn, \mathcal{Y} has of course a maximal integrable element for the a.s. order (see e.g. Lemma 4.3 in [25]) and therefore the previous result does not preclude reflexivity in that case. Indeed, in the complete case and under Assumption 2.2, one obtains from the proof of Theorem 2.4 that L_I is a reflexive Orlicz-Musielak space and

so, owing to the existence of a saddle point, one can be more specific about the solution to the robust optimization problem. This is done in Theorems 2.2.5 and Section 2.4.1 of the thesis [4], whereby the author relates the dual and primal optimizers as expected from e.g. [39]. However, this is not the point about complete markets we want to stress in this article; our main contribution in such case is that, thanks to our functional analytical approach and the nice Orlicz space structure it leads to in the complete setting, we are able to provide in a systematic way the characterization of the saddle-point element $\hat{\mathbb{Q}}$, that is, the *worst* element in \mathcal{Q} for each utility function.

Before detailing our specific results for complete markets in Section 2.3, let us discuss our assumptions and the relationship between Theorem 2.4 and results in the existing literature.

2.2 Discussion on Assumption 2.3 and comparison to the existing literature

On the utility function: Condition $U(0+) = 0$ was assumed as it implies the desirable property $V \geq 0$. If U were bounded from below our results would still hold. The following are examples of utility functions for which our results apply:

Example 2.6. Power utilities $U(\cdot) = \alpha^{-1}(\cdot)^\alpha$, $\alpha \in (0, 1)$ fulfill point 1. in Assumption 2.3. Moreover, this assumption is satisfied if and only if U^{-1} is convex and increasing, $U^{-1}(0+) = 0$, $[U^{-1}]'(0+) = 0$ and $[U^{-1}]'(\lim_{x \rightarrow \infty} U(x)) = \infty$. So for instance the inverse on $[0, +\infty)$ of $x \mapsto e^x - x - 1$ satisfies it as well. Power utilities, as described, also satisfy Assumption 2.2.

On the non-emptiness of \mathcal{Y}^* The reason behind point 2. of Assumption 2.3 is two-fold. It ensures that the result in Remark 3.12 stating that the Orlicz-Musielak spaces $L_{|\cdot|V \circ Y/|\cdot|}$ and $L_{YU^{-1} \circ |\cdot|}$ are well-behaved whenever $Y \in \mathcal{Y}^*$, be lifted to the spaces L_I and L_J . On the other hand, it precludes the combinations of market models and utility functions for which, even in the non-robust case, primal optimizers do not exist; we come back to this under the point “Global comparison of our assumptions”.

On the topological constraint on \mathcal{Q} : Our point 3.(c) in Assumption 2.3, specifically $d\mathcal{Q}/d\mathbb{P} \subset L_I$, implies that $\forall \mathbb{Q} \in \mathcal{Q}, \exists y > 0, v_{\mathbb{Q}}(y) < \infty$. Under the assumption $L_I^* \cong L_J$ in our Theorem 2.4, we further have that $\forall \mathbb{Q} \in \mathcal{Q}, \forall y > 0, v_{\mathbb{Q}}(y) < \infty$. This is a strengthening of Condition (2.10) in [39], which the authors there use to prove existence of optimal wealth processes. Furthermore, in Theorem 2.2. of [39] the authors succeed in proving conjugacy of the value functions without anything like our condition $d\mathcal{Q}/d\mathbb{P} \subset L_I$, but in turn suppose the stronger L^0 -closedness condition, typically assumed in the literature (see e.g. [39], [17]) and equivalent in the present context to weak L^1 compactness. By Proposition 5.8 below, our weak-closedness condition for $d\mathcal{Q}/d\mathbb{P}$ is indeed implied by the usual closedness in L^0 and, as the following example exhibits, the converse is not true. Furthermore, the same example shows that in our setting a “least-favourable” measure might not exist, contrary to the framework of [6], [38]:

Example 2.7. Assume an investor knows or anticipates that the mean of a \mathcal{F}_T -measurable unbounded random variable h (e.g. $h = S_T$) is bounded from below by a constant $A > 0$. If $\mathbb{E}(h) < \infty$, then the set of densities $\frac{d\mathbb{Q}_A}{d\mathbb{P}}$ of the set $\mathcal{Q}_A := \{\mathbb{Q} \in \mathcal{P} : \mathbb{Q} \ll \mathbb{P}, \mathbb{E}^{\mathbb{Q}}(h) \geq A\}$ is not closed in L^0 . Indeed, the sequence $\mathbb{Q}^n(\cdot) := \mathbb{P}(\cdot | h \geq nA) \in \mathcal{Q}_A$, is such that $\frac{d\mathbb{Q}^n}{d\mathbb{P}} = \mathbb{P}(h \geq nA)^{-1} \mathbb{1}_{\{h \geq nA\}} \rightarrow 0$ in L^0 when $n \rightarrow \infty$, yet obviously $0 \notin \mathcal{Q}_A$. Consider now the utility function $U(x) = \frac{x^\alpha}{\alpha}$, $\alpha \in (0, 1)$, so that after some computations

we see $L_{|\cdot|V \circ 1/|\cdot|} = L^{\frac{1}{\alpha}}$, and call $\tilde{\mathcal{Q}}_A := \{\mathbb{Q} \in \mathcal{P} : \mathbb{Q} \ll \mathbb{P}, d\mathbb{Q}/d\mathbb{P} \in L^{\frac{1}{\alpha}}, \mathbb{E}^{\mathbb{Q}}(h) \geq A\}$, which by the same argument is not closed in L^0 . If however h is an element of $L^{\frac{1}{1-\alpha}}$, one can check that $\tilde{\mathcal{Q}}_A$ is a closed subset of $L_{|\cdot|V \circ 1/|\cdot|}$. Finally, it is not difficult to see with the aid of Lagrange multipliers and under given conditions on h and A , that the solution of $\inf\{\mathbb{E}[Z^2] : Z \in d\tilde{\mathcal{Q}}_A/d\mathbb{P}\}$ is a linear function of h , whereas the solution of $\inf\{\mathbb{E}[Z^{3/2}] : Z \in d\tilde{\mathcal{Q}}_A/d\mathbb{P}\}$ is a quadratic function of h . In particular then $\tilde{\mathcal{Q}}_A$ has no least-favourable measure in the sense of e.g. [6], [38], [18].

Finally, since we cannot get countable convexity out of convexity with our weak-closure assumption, this condition has been put in Assumption 2.3. Had we required $\mathbb{Q}_e \neq \emptyset$ instead, we could have assumed usual convexity. Condition 3.(d) therein, which we add straight from the beginning, is required in any case for all the results in the literature.

Remark 2.8. Our motivation for the set \mathcal{Q} comes from modeling concerns, namely we want to account for anticipation of moments of observables or insider information of statistical kind, rather than from axiomatic considerations. Regarding these, one could consider $Z \in L_J \mapsto \rho_{\mathcal{Q}} := -\inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}}[-Z]$ as a coherent risk measure and survey its properties in terms of conditions on \mathcal{Q} , and conversely ask which coherent risk measures on L_J can be obtained as $\rho_{\mathcal{Q}}$ for \mathcal{Q} in some appealing class. In general the dual representation of a coherent risk measure via a not necessarily compact set is only equivalent to lower semicontinuity, and for nice lattices as our Modular spaces, to the Fatou property; see [9] on the so-called C-property, which holds in our setting. If L_J were an Orlicz heart, one can find in [34, Corollary 7] (resp. [12, Corollary 4.1]) the equivalence between continuity from below of $\rho_{\mathcal{Q}}$ (resp. Lipschitz continuity) and weak compactness (resp. norm boundedness) in the dual space of $d\mathcal{Q}/d\mathbb{P}$. Along similar lines, one could ask what kind of preferences can be numerically represented via “expected utility under multiple priors”, as in [19] or [30], and further investigate the properties of this representations in terms of the set of priors \mathcal{Q} .

Global comparison of our assumptions: Comparing our results with those in [39] and [17], which we take as benchmark, we find that we must require stronger integrability of the elements in $d\mathcal{Q}/d\mathbb{P}$, better-behaved utility functions, and the more stringent dual finiteness condition $\mathcal{Y}^* \neq \emptyset$. If the set \mathcal{Q} were closed in L^0 our assumptions would be then unnecessarily demanding. The point is, we prove min-max equality and duality of value functions beyond the L^0 -closedness assumption.

We stress that Theorem 2.4 can be applied to the classical, non-robust situation as well, asserting the existence of an optimal wealth process and the finiteness everywhere of the value function, under the assumptions that U is a utility function on $(0, \infty)$ bounded from below and satisfying INADA, and that $\mathcal{Y}^* \neq \emptyset$ (see (2.8)). These conditions are not necessary for the existence of optimal wealths, and as sufficient conditions they are stronger than the one given in [26] (namely finiteness of the dual value function). The point is, our modular space proof is purely functional-analytical (see Proposition 2.5.6 in [4] for a self-contained proof not relying on Theorem 2.4) and the condition on \mathcal{Y}^* precludes market models as in Example 5.2 in [25], whereby the failure of $AE(U) < 1$ implies non-existence of optimal wealths. This functional-analytical proof is only seemingly shorter or neater than the classical one (relying in convex-compactness of the solid hull of $\mathcal{X}(1)$), since it relies on the fact that L_J is a norm-dual space (see Proposition 5.9), which is lengthy to prove. Even taking this fact from granted, one still needs to use the bipolar theorem, which is in any case necessary to prove the mentioned property of the solid hull of $\mathcal{X}(1)$.

Related use of Orlicz spaces in the literature: We compare our Modular space

approach (Orlicz-Musiela space, in case of complete markets) to that of [17,21], which to the best of our knowledge are the only works where Orlicz spaces are used for the problem of robust utility maximization. In their setting the set \mathcal{Q} is assumed weakly compact in L^1 and therefore, out of Valle-Poussin criterion, it is bounded in an Orlicz space induced by a well-behaved Young function. Starting from this the authors eventually prove that the set of martingale measure densities possibly contributing to the dual problem is also weakly compact in L^1 , by means of constructing a second Young function. This establishes full dual attainability under the hypotheses given. We in turn look for compactness elsewhere, in the set of images of the terminal wealths through the utility function, and do not ask for compactness of \mathcal{Q} . We do establish that the relevant elements for the dual problem, say $\tilde{\mathcal{Q}} \subset \mathcal{Q}$, are bounded in the modular space (see Remark 5.3), but this is a long way from implying that they form a weakly compact set in L^1 . It is only for complete markets under Assumption 2.2 and under the condition that \mathbb{P} is a martingale measure, that our spaces become faintly comparable to those of [17]. In such case, as the proof of Lemma 2.11 or Remark 2.13 in [17] reveals (taking $f = g$ therein), the classical Orlicz space $L_{|\cdot|V \circ 1/|\cdot|}$ we get has a strictly stronger topology than any of the Orlicz spaces introduced there, and cannot be expressed in terms of any of them.

2.3 Main results in the complete case: characterization of the worst-case measure

If in the complete case we denote by Y^* the density of the unique equivalent martingale measure w.r.t. \mathbb{P} , it is easy to see that the spaces L_I and $L_{|\cdot|V \circ Y^*/|\cdot|}$, defined in Section 2.1, must coincide. We may thus assume without loss of generality that the reference measure \mathbb{P} , whose single role is to specify null sets, is a martingale measure and so $Y^* \equiv 1$ (this does not trivialize our robust problem as measures in \mathcal{Q} need not to be martingale measures even if \mathbb{P} is). Under this assumption, Lemma 4.3 in [25] and its proof states that every terminal value of the elements $Y \in \mathcal{Y}_{\mathbb{P}}(1)$ is bounded by 1 and (since V is non-increasing) we have:

$$v(y) = \inf_{Z \in \frac{d\mathcal{Q}}{d\mathbb{P}}} \mathbb{E} \left[ZV \left(\frac{y}{Z} \right) \right]. \quad (2.13)$$

By the same argument, the functions space L_I we worked with in Section 2.1 is $L_{|\cdot|V \circ 1/|\cdot|}$; a classical Orlicz space. The space L_J corresponds accordingly to the Orlicz space $L_{U^{-1} \circ |\cdot|}$. All in all, we can write:

$$v(y) = \inf \left\{ \mathbb{E}(\gamma_y^*(Z)) : Z \in L_{|\cdot|V \circ 1/|\cdot|} \text{ s.t. } \mathbb{E}^{\mathbb{P}}(Z) = 1 \text{ and } Z \cdot d\mathbb{P} \in \mathcal{Q} \right\}, \quad (2.14)$$

where $\gamma_y^*(\cdot)$ is the convex function which equals $z \mapsto zV\left(\frac{y}{z}\right)$ for $z > 0$ and $+\infty$ otherwise (see Lemma 3.9 for further properties of this γ_y^*).

We will assume that the set \mathcal{Q} is defined by the constraint that the under each element $\mathbb{Q} \in \mathcal{Q}$, the average value of a given observable of the market θ , with values in a (possibly infinite dimensional) vector space, lies on a prescribed convex subset of this space. More precisely, we will consider

- i) $(\mathbf{F}_0, \mathbf{G}_0)$ a pair of linear spaces of arbitrary dimension, with \mathbf{F}_0 the algebraic dual of \mathbf{G}_0 and with dual product denoted $\langle \cdot, \cdot \rangle_{\mathbf{G}_0, \mathbf{F}_0}$.
- ii) $\theta : \Omega \rightarrow \mathbf{F}_0$ a function (or “observable”) on the market with values in \mathbf{F}_0 and
- iii) $\mathbf{C}_0 \subset \mathbf{F}_0$ a convex subset .

In this setting, we will characterize the worst-case measure $\hat{\mathbb{Q}}$ using techniques for the minimization of abstract entropy functionals, developed in the series of papers [27–29]. Following those works we will make

Assumption 2.9.

- i) $\forall g \in \mathbf{G}_0$, the function $\omega \in \Omega \mapsto \langle g, \theta(\omega) \rangle_{\mathbf{G}_0, \mathbf{F}_0}$ is measurable.
- ii) $\forall g \in \mathbf{G}_0$, $\langle g, \theta \rangle_{\mathbf{G}_0, \mathbf{F}_0} \in L_{U^{-1} \circ |\cdot|}$.
- iii) $\forall (g, a) \in \mathbf{G}_0 \times \mathbb{R}$, one has $\langle g, \theta(\cdot) \rangle_{\mathbf{G}_0, \mathbf{F}_0} = a$ \mathbb{P} -a.s. iff $g = 0$ and $a = 0$.
- iv) We have $\mathcal{Q} \neq \emptyset$ and

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \left\{ Z \in L_{|\cdot|V \circ 1/|\cdot|} : Z \geq 0 \text{ a.s., } \mathbb{E}(Z) = 1 \text{ and } \Theta(Z) \in \mathbf{C}_0 \right\},$$

where $\Theta : L_{|\cdot|V \circ 1/|\cdot|} \rightarrow \mathbf{F}_0$ denotes the linear operator $\Theta(Z) = \int \theta Z d\mathbb{P}$ such that

$$\langle g, \Theta(Z) \rangle_{\mathbf{G}_0, \mathbf{F}_0} = \int_{\Omega} \langle g, \theta \rangle_{\mathbf{G}_0, \mathbf{F}_0} Z d\mathbb{P}, \quad g \in \mathbf{G}_0.$$

- iv) The market is complete and Assumption 2.2 holds, so $L_{|\cdot|V \circ 1/|\cdot|}$ is reflexive
- v) Assumption 2.3 holds, in particular $\frac{d\mathcal{Q}}{d\mathbb{P}} \subset L_{|\cdot|V \circ 1/|\cdot|}$ is $\sigma(L_{|\cdot|V \circ 1/|\cdot|}, L_{U^{-1} \circ |\cdot|})$ -closed.

We observe that if point (c) of Assumption 2.3 holds and Assumption 2.2 on U is enforced, then point iv) of Assumption 2.9 is satisfied and one can write

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \bigcap_{\lambda \in \Lambda} \left\{ \frac{d\mathbb{Q}}{d\mathbb{P}} : \frac{d\mathbb{Q}}{d\mathbb{P}} \in L_{|\cdot|V \circ 1/|\cdot|} \text{ and } \mathbb{E}^{\mathbb{Q}}(h_{\lambda}) \in [a_{\lambda}, \infty) \right\}$$

for some family $(h_{\lambda})_{\lambda \in \Lambda}$ of elements of $L_{U^{-1} \circ |\cdot|}$ and some $a = (a_{\lambda})_{\lambda \in \Lambda} \in \mathbb{R}^{\Lambda}$, by Hahn-Banach Theorem. This grants that points i) and ii) of Assumption 2.9 hold with $\mathbf{F}_0 = \mathbb{R}^{\Lambda}$, $\mathbf{G}_0 = \bigoplus_{\lambda \in \Lambda} \mathbb{R}$, $\theta(\omega) = (h_{\lambda}(\omega))_{\lambda \in \Lambda}$ and $\mathbf{C}_0 = \prod_{\lambda \in \Lambda} [a_{\lambda}, \infty)$; point iii) holds if the family $(h_{\lambda})_{\lambda \in \Lambda} \cup \{1\}$ is linearly independent, or otherwise can be obtained by replacing \mathbf{G}_0 by a suitable quotient space (see Section 4 for details or the Example 2.12 below for a concrete instance). Assumption 2.9 is not an actual restriction, in the setting of Theorem 2.4 in complete markets. In turn, it allows us to deal with uncertainty sets naturally arising in modeling situations, for instance market information specified by moments of observables, the probability of a given event, or even the full law of a given observable or the flow of time-marginal laws of a random process (as considered e.g. in [6]). See Examples 2.11 to 2.14 below in this section.

In order to characterize the minimizing density $\hat{Z} = \frac{d\hat{\mathbb{Q}}}{d\mathbb{P}}$ in (2.14) and following [27–29], we will formulate a dual problem to it in some space \mathbf{G} , solvable under some weak qualification condition. To that end notice that, as a consequence of points i), ii) and iii) of Assumption 2.9 (see Section 4 for additional discussion), the mapping

$$(g_0, a) \in \mathbf{G}_0 \times \mathbb{R} \mapsto \langle g_0, \theta(\cdot) \rangle_{\mathbf{G}_0, \mathbf{F}_0} + a \in L_{U^{-1} \circ |\cdot|}$$

embeds the space $\mathbf{G}_1 := \mathbf{G}_0 \times \mathbb{R}$ into $L_{U^{-1} \circ |\cdot|}$ and thus induces a norm on \mathbf{G}_1 . We denote by \mathbf{G} the completion of \mathbf{G}_1 under it, which is isomorphic to a closed subspace of $L_{U^{-1} \circ |\cdot|}$, and call \mathbf{F} the topological dual of \mathbf{G} , which is a linear subspace of $\mathbf{F}_1 := \mathbf{F}_0 \times \mathbb{R}$. Write $\langle g, f \rangle$ for the natural dual product of the pair $(g, f) \in \mathbf{G} \times \mathbf{F}$ and denote by

$$\langle g, \theta_1 \rangle \in L_{U^{-1} \circ |\cdot|}$$

the element identified with $g \in \mathbf{G}$; in particular, $\langle g, \theta_1 \rangle = \langle g_0, \theta \rangle_{\mathbf{G}_0, \mathbf{F}_0} + \beta \cdot 1$ if $g = (g_0, \beta) \in \mathbf{G}_1$. Setting $\mathbf{C}_1 := (\mathbf{C}_0 \times \{1\})$ and $\mathbf{C} := \mathbf{C}_1 \cap \mathbf{F}$, the dual problem of (2.14) is:

$$\text{Maximize } \inf_{f \in \mathbf{C}} \langle g, f \rangle - y \mathbb{E} [U^{-1} ((\langle g, \theta_1 \rangle)_+)] , \quad g \in \mathbf{G}. \quad (2.15)$$

The following functional

$$\Gamma_y^*(f, s) := \sup_{g \in \mathbf{G}_0} \sup_{\beta \in \mathbb{R}} \langle g, f \rangle_{\mathbf{G}_0, \mathbf{F}_0} + s\beta - y \mathbb{E} [U^{-1} ((\beta + \langle g, \theta \rangle_{\mathbf{G}_0, \mathbf{F}_0})_+)] , \quad (f, s) \in \mathbf{F}_1 \quad (2.16)$$

will be useful to state sufficient conditions for primal-dual equality between the pair of problems (2.14) and (2.15). Moreover, we will also state in terms of it a weak qualification condition ensuring dual attainability and allowing us to characterize the solution of (2.14). Recall that the *affine hull* $\text{aff}(A)$ of $A \subset L$, where L is a linear space, is the smallest affine subspace of L containing A , and the *intrinsic core* of A , given by

$$\text{icor}(A) := \{a \in A | \forall x \in \text{aff}(A), \exists t > 0 \text{ st. } a + t(x - a) \in A\} ,$$

is the largest topology-free notion of its *interior*. Our main result in the complete case is:

Theorem 2.10. *Suppose that Assumption 2.9 holds.*

a) *For each $y > 0$, the following identities hold:*

$$\begin{aligned} v(y) &= \inf_{f \in \mathbf{C}_0} \Gamma_y^*(f, 1) \\ &= \sup_{g \in \mathbf{G}_0} \sup_{\beta \in \mathbb{R}} \left(\inf_{f \in \mathbf{C}_0} \langle g, f \rangle_{\mathbf{G}_0, \mathbf{F}_0} + \beta \right) - y \mathbb{E} [U^{-1} ((\beta + \langle g, \theta \rangle_{\mathbf{G}_0, \mathbf{F}_0})_+)] \\ &= \sup_{g \in \mathbf{G}} \left(\inf_{f \in \mathbf{C}} \langle g, f \rangle \right) - y [U^{-1} ((\langle g, \theta_1 \rangle)_+)] . \end{aligned} \quad (2.17)$$

Moreover, if $\mathbf{C}_1 \cap \text{dom}(\Gamma_y^*) \neq \emptyset$ then the infimum in (2.14) is attained at a unique element $Z^y \in \frac{dQ}{d\mathbb{P}}$ and the four expressions in (2.17) equal $\mathbb{E}(\gamma_y^*(Z^y))$. If in addition $\mathbf{C}_1 \cap \text{icor}(\text{dom}(\Gamma_y^*)) \neq \emptyset$, then problem (2.15) has a solution $g \in \mathbf{G}$.

b) *A pair $(Z^y, g^y) \in L_{|\cdot|V \circ 1/|\cdot|} \times \mathbf{G}$ solves problems (2.14) and (2.15) if and only if*

$$\begin{cases} \bullet & (\Theta(Z^y), 1) \in \mathbf{C} \cap \text{dom } \Gamma_y^* , \\ \bullet & \langle g^y, (\Theta(Z^y), 1) \rangle \leq \langle g^y, f \rangle \text{ for all } f \in \mathbf{C} \cap \text{dom } \Gamma_y^* \text{ and} \\ \bullet & Z^y = y [U^{-1}]' ((\langle g^y, \theta_1 \rangle)_+). \end{cases} \quad (2.18)$$

In particular, $(\langle g^y, \theta_1 \rangle)_+ = (\langle \hat{g}^y, \theta_1 \rangle)_+$, \mathbb{P} -a.s. for any solutions $g^y, \hat{g}^y \in \mathbf{G}$ to (2.15).

c) *If $\mathbf{C}_1 \cap \text{icor}(\text{dom}(\Gamma_y^*)) \neq \emptyset$ for all $y > 0$, then for all $x > 0$, we have:*

$$u(x) = \inf_{y > 0} \left(\inf_{f \in \mathbf{C}_0} \Gamma_y^*(f, 1) + xy \right) = \inf_{y > 0} \left(\mathbb{E} \left[Z^y V \left(\frac{y}{Z^y} \right) \right] + xy \right) = \mathbb{E} \left[Z^{\hat{y}} V \left(\frac{\hat{y}}{Z^{\hat{y}}} \right) \right] + x\hat{y},$$

where \hat{y} belongs to the super-differential of u at x .

Thus, in a complete market and under the assumptions of Theorem 2.10, finding the worst-case measure $\hat{\mathbb{Q}}$ attaining the infimum in (2.12) amounts to first finding for each $y > 0$ a solution g^y to (2.15) and computing $v(y) = \mathbb{E} [Z^y V (\frac{y}{Z^y})]$, where $Z^y =$

$y[U^{-1}]'((\langle g^y, \theta_1 \rangle)_+)$, then finding $\hat{y} > 0$ that minimizes the obtained values of $v(y) + xy$ and setting $\hat{\mathbb{Q}} = Z^{\hat{y}} \cdot \mathbb{P}$.

For each $y > 0$, problem (2.15) dual to (2.14) is, in a way, a “dual of a dual problem” to the original problem (2.4). The difference is that the first dualization is w.r.t. the budget constraint whereas the second one is w.r.t. the constraints determining the uncertainty set. The assumption $\mathbf{C}_1 \cap \text{icor}(\text{dom}(\Gamma_y^*)) \neq \emptyset$ corresponds to a constraint qualification condition of geometric (rather than topological) type for the last dualization. Note that in many practical instances, problem (2.15) can be finite-dimensional:

Example 2.11. Consider $S_t = \exp\left\{-\frac{\sigma^2}{2}t + \sigma W_t\right\}$ the risk-neutral Samuelson-Black-Scholes model, with W a standard Brownian motion, $\sigma^2 > 0$ and $S_0 = 1$ (for simplicity). We take $U(x) = 2x^{1/2}$ and $\mathcal{Q}_A := \{\mathbb{Q} \in \mathcal{P} : \mathbb{Q} \ll \mathbb{P}, \mathbb{E}^{\mathbb{Q}}(S_T) \geq A, d\mathbb{Q}/d\mathbb{P} \in L^2\}$ for fixed $A > 0$, so that $L_{|\cdot|V \circ 1/|\cdot|} = L_{U^{-1} \circ |\cdot|} = L^2$ and $\frac{d\mathbb{Q}_A}{d\mathbb{P}}$ is weakly closed in L^2 , by Example 2.7 with $h := S_T$. Girsanov Theorem yields for each $A > 0$ the existence of a probability measure \mathbb{Q}_A s.t. $\frac{d\mathbb{Q}_A}{d\mathbb{P}} \in L^2$ and $\mathbb{E}^{\mathbb{Q}_A}(S_T) = A$, hence $\mathcal{Q}_A \neq \emptyset$. Moreover, \mathcal{Q}_A is closed under infinite convex combinations, and since S_T and 1 are obviously linearly independent r.v., Assumption 2.9 holds.

Furthermore, we can directly check that $\Theta_1(\frac{d\mathbb{Q}_A}{d\mathbb{P}}) \in \text{dom}(\Gamma_y^*)$ (or alternatively use the “little dual equality” (4.5)) in order to get that $\mathbf{C}_1 \cap \text{dom}(\Gamma_y^*) \neq \emptyset$. Since for any $(a, b) \in \mathbb{R}_+^2$ with $a, b \neq 0$ there is $Z \in L^2, Z \geq 0$ such that $(\mathbb{E}(Z), \mathbb{E}(ZS_T)) = (a, b)$ (take e.g. $Z := a\frac{d\mathbb{Q}_A}{d\mathbb{P}} \in L^2$ with \mathbb{Q}_A as above with $A = \frac{b}{a}$) we similarly check that $\mathbf{C}_1 \cap \text{icor}(\text{dom}(\Gamma_y^*)) \neq \emptyset$.

We next solve the (second) maximization problem in (2.17), that is

$$\begin{aligned} \sup_{(\beta, \alpha)} \left[\inf_{c \geq A} \beta + c\alpha - \mathbb{E}^{\mathbb{P}}(yU^{-1}(\beta + S_T\alpha)_+) \right] &= \sup_{\beta \in \mathbb{R}, \alpha \geq 0} [\beta + A\alpha - \mathbb{E}^{\mathbb{P}}(yU^{-1}(\beta + S_T\alpha)_+)] \\ &= \sup_{\beta \in \mathbb{R}, \alpha \geq 0} \beta + A\alpha - \frac{y}{4} \mathbb{E}^{\mathbb{P}}((\beta + S_T\alpha)^2 \mathbf{1}_{\beta + S_T\alpha > 0}) \end{aligned}$$

In order to get explicit expressions, we assume that $e^{\sigma^2 T} > A > 1$. Upon explicitly computing the expectation, we notice that, in that case, the unique critical point of the above concave function is $g^y = \left(\frac{2(e^{\sigma^2 T} - A)}{y(e^{\sigma^2 T} - 1)}, \frac{2(A-1)}{y(e^{\sigma^2 T} - 1)} \right) \in (0, \infty)^2$, with optimal value $\frac{1}{y} \left[1 + \frac{(A-1)^2}{e^{\sigma^2 T} - 1} \right]$ (see Example 2.2.3 in [4] for details). We deduce that

$$u(x) = 2\sqrt{x \left(1 + \frac{(A-1)^2}{e^{\sigma^2 T} - 1} \right)}, \quad \hat{\mathbb{Q}}(d\omega) := \frac{e^{\sigma^2 T} - A + S_T(A-1)}{e^{\sigma^2 T} - 1} \mathbb{P}(d\omega).$$

Hence $\hat{\mathbb{Q}}$ is the unique convex combination of the measures \mathbb{P} and $S_T \cdot \mathbb{P}$ which is a probability measure and satisfies $\mathbb{E}^{\hat{\mathbb{Q}}}(S_T) = A$. Classic results in the non-robust setting (cf. [25]) yield

$$\hat{X}_T := x \frac{\left(e^{\sigma^2 T} - A + S_T(A-1) \right)^2}{(e^{\sigma^2 T} - 1 + (A-1)^2)(e^{\sigma^2 T} - 1)}, \mathbb{P} \text{ and } \hat{\mathbb{Q}} \text{ a.s.}$$

and the robust optimal strategy can then be derived by standard hedging arguments.

Example 2.12. Let (E, Σ) be measurable space and $\vartheta : \Omega \rightarrow E$ a measurable “observable” of the market. Let $\nu \ll \mu := \mathbb{P} \circ \vartheta^{-1}$ be a probability measure on E with $\frac{d\nu}{d\mu} \in L_{|\cdot|V \circ 1/|\cdot|}(E, \Sigma, \mu)$ and assume

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \left\{ Z \in L_{|\cdot|V \circ 1/|\cdot|} : \mathbb{E}(Z) = 1, Z \geq 0 \text{ a.s. and } (Z \cdot \mathbb{P}) \circ \vartheta^{-1} = \nu \right\}.$$

Taking $\mathbf{G}_0 = \mathcal{B}/\mathcal{B}^\mathbb{P}$ with $\mathcal{B} := \{g : E \rightarrow \mathbb{R}, \text{ bounded measurable}\}$ and $\mathcal{B}^\mathbb{P} := \{g \in \mathcal{B} : g = \text{cst.} \mu - \text{a.s.}\}$, $\mathbf{F}_0 = \{f : \Sigma \rightarrow \mathbb{R} : f \text{ finite signed measure}\}$ with $\langle g + \mathcal{B}^\mathbb{P}, f \rangle_{\mathbf{G}_0, \mathbf{F}_0} := \int g(x)f(dx)$ and $\theta(\omega) = \delta_{\vartheta(\omega)}$, points i) to iv) of Assumption 2.9 hold. Also, problem (2.15) is equivalent to

$$\text{Maximize } \mathbb{E} \left[g(\vartheta) \frac{d\nu}{d\mu}(\vartheta) - y U^{-1}(g_+(\vartheta)) \right], \quad g \in L_{U^{-1} \circ |\cdot|}(E, \Sigma, \mu),$$

The first order optimality condition for this problem is

$$\mathbb{E} \left[g(\vartheta) \left(\frac{d\nu}{d\mu}(\vartheta) - y [U^{-1}]' \left(g_+^{y,U}(\vartheta) \right) \right) \right] = 0,$$

for all bounded measurable $g : E \rightarrow \mathbb{R}$. From this and part b) of Theorem 2.10 we get that, provided we can always find $g^{y,U}$ such that $y [U^{-1}]' \left(g_+^{y,U}(\vartheta) \right) = \frac{d\nu}{d\mu}(\vartheta)$, \mathbb{P} -a.s., then the primal solution is Z independent on y and U , and so we recover the least-favourable measure found in [6]. It is clear we can indeed find such $g^{y,U} \in \mathbf{G}$ in this case.

Remark 2.13. Example 2.12 points out to a more general result. Indeed, it is not difficult to see that if $\mathbf{C} = \{f\}$ is a singleton, which by Hahn-Banach can be associated to a unique minimal representative $Z \in L_{|\cdot|V \circ 1/|\cdot|}$ characterized by $\langle g, f \rangle = \mathbb{E}[Z \langle g, \theta_1 \rangle]$ and its measurability w.r.t. the sigma-field generated by $\{\langle g, \theta_1 \rangle : g \in \mathbf{G}\}$ (see the beginning of Section 4.1), then provided $Z = y[U^{-1}]'(\langle g^{y,U}, \theta_1 \rangle_+)$ is always solvable we get as before that Z is the primal solution and this is independent of y and U . We roughly conjecture in the complete case that the existence of a least-favourable measure in \mathcal{Q} is related to the properties that, for every nice Young function ϕ^* such that the associated Orlicz space is reflexive and contains $d\mathcal{Q}/d\mathbb{P}$, the set \mathcal{Q} can be written down with $\mathbf{C} = \{f\} \subset \mathbf{F}$ a singleton and that the minimal representative Z satisfies $[\phi']^{-1}(Z) \in \{\langle g, \theta_1 \rangle_+ : g \in \mathbf{G}\}$.

Example 2.14. Assume $\vartheta = (\vartheta_t)_{t \in [0, T]}$ is under \mathbb{P} a continuous process with values in \mathbb{R}^d and

$$\frac{d\mathcal{Q}}{d\mathbb{P}} = \left\{ Z \in L_{|\cdot|V \circ 1/|\cdot|} : \mathbb{E}(Z) = 1, Z \geq 0 \text{ a.s. and } (Z \cdot \mathbb{P}) \circ \vartheta_t^{-1} = \nu_t, t \in [0, T] \right\}$$

for a flow of probability laws $(\nu_t)_{t \in [0, T]}$ s.t. $\nu_t \ll \mathbb{P} \circ \vartheta_t^{-1}$ (as succinctly studied in [6]). We can take $\mathbf{G}_0 = \{g \in C([0, T] \times \mathbb{R}^d, \mathbb{R}) : \text{vanishing when } |x| \rightarrow \infty\}$, $\mathbf{F}_0 = C([0, T], \mathcal{M}(\mathbb{R}^d))$, where $\mathcal{M}(\mathbb{R}^d)$ is the space of finite signed measures on \mathbb{R}^d endowed with the weak topology, $\langle g, f \rangle_{\mathbf{G}_0, \mathbf{F}_0} := \int_0^T \int_{\mathbb{R}^d} g(t, x) f_t(dx) dt$ and $\theta = (\delta_{\vartheta_t})_{t \in [0, T]}$. The validity of Assumption 2.9 and the solvability of problem (2.15) will in general depend on the market and on ϑ , and can be studied in specific instances (this is work in progress, but see [3, Chapter 3.6.2] in Spanish).

3 Orlicz-Musielak spaces and the robust optimization problem

We now introduce some general functional spaces needed in our study of the robust optimization problem. These can actually be seen as Orlicz spaces based on “randomized Young functions”. Their main properties including dual spaces and reflexivity are first reviewed in Section 3.1, following succinctly the presentation in [23, 24]. Then in Section 3.2 we translate and apply these concepts to the robust optimization setting, for which some relevant functionals are introduced and a few technical results are established.

3.1 Orlicz-Musielak Spaces

Recall that $(\Omega, \mathcal{F}, \mathbb{P})$ is a (complete) probability space and that the notation $\mathbb{E}(\cdot)$ is employed for the expectation under \mathbb{P} .

Definition 3.1. A functional $\rho : \mathbb{R} \times \Omega \rightarrow [0, \infty]$ is said to be a rho-functional if the following hold:

1. ρ is jointly measurable
2. for almost every $\omega \in \Omega$, $\rho(\cdot, \omega)$ is lower-semicontinuous and convex
3. $\rho(0, \cdot) \equiv 0$ and $\rho(x, \cdot) = \rho(-x, \cdot)$
4. If $\alpha : \Omega \rightarrow (0, \infty)$ is measurable, then there exists a measurable function $\lambda : \Omega \rightarrow (0, \infty)$ such that a.s. $[|x| \geq \lambda(\omega) \Rightarrow \rho(x, \omega) \geq \alpha(\omega)]$.
5. If $\epsilon : \Omega \rightarrow (0, \infty)$ is measurable, then there exists a measurable function $\rho : \Omega \rightarrow (0, \infty)$ such that a.s. $[|x| \leq \rho(\omega) \Rightarrow \rho(x, \omega) \leq \epsilon(\omega)]$.
6. The random variables $\rho(x, \cdot)$ and $\rho^*(y, \cdot) := \sup_{x \in (-\infty, \infty)}(xy - \rho(x, \cdot))$ are integrable for every $x, y \in (-\infty, \infty)$.

Remark 3.2. Under the conditions in Definition 3.1, the results in [23] are valid. It is worth noting that in that paper a functional ρ satisfying conditions 1. through 5. was called an “N-function”. However, such a ρ “only” converges a.s. to zero (resp. to ∞) when x tends to zero (resp. to ∞), whereas in the nowadays standard definition of N-functions, it is the quotient $\frac{\rho(x, \omega)}{x}$ that has this limiting behaviour in x near 0 and $+\infty$. To avoid confusions we use here the different “rho-functional” terminology. Also, we note that in the language of [23], the above condition 6. amounts to requiring “condition B on ρ and ρ^* ”, and is necessary to obtain topological duality results. Last, it is not difficult to see from the above definition that ρ^* is also a rho-functional.

Define now for a random variable $Z : \Omega \rightarrow (-\infty, \infty)$,

$$I_\rho(Z) := \mathbb{E}[\rho(Z, \cdot)] \leq \infty$$

In the terminology of [23], this is a normal convex modular. This allows us to define the following spaces:

Definition 3.3. The Orlicz-Musielak space associated to ρ is defined as:

$$L_\rho(\Omega, \mathbb{P}) := \{Z \in L^0 \text{ s.t. } \exists \alpha > 0, I_\rho(\alpha Z) < \infty\}, \quad (3.1)$$

and its *Orlicz heart* is the subspace:

$$E_\rho(\Omega, \mathbb{P}) := \{Z \in L^0 \text{ s.t. } \forall \alpha > 0, I_\rho(\alpha Z) < \infty\}. \quad (3.2)$$

In the following, L_ρ will stand as an abbreviation for $L_\rho(\Omega, \mathbb{P})$. The following result is a compendium of known facts; see Theorem 2.3.1 in [4] for the references:

Theorem 3.4. *The following functionals define equivalent norms on L_ρ :*

$$\|Z\|_\rho^l := \inf \left\{ \beta > 0 : I_\rho \left(\frac{Z}{\beta} \right) \leq 1 \right\}, \quad (3.3)$$

$$\|Z\|_\rho^a := \sup \left\{ \mathbb{E}(\phi Z) : \phi \in L_{\rho^*}, \hat{I}_\rho(\phi) \leq 1 \right\} \quad (3.4)$$

$$= \sup \left\{ \mathbb{E}(\phi Z) : \phi \in L_{\rho^*}, \|\phi\|_{\rho^*}^l \leq 1 \right\}, \quad (3.5)$$

where $\hat{I}_\rho(\phi) := \sup_{Z \in L_\rho} [\mathbb{E}(\phi Z) - I_\rho(Z)] = I_{\rho^*}$, and $\rho^*(\cdot, \omega)$ is the a.s. convex conjugate of $\rho(\cdot, \omega)$ as defined previously. Moreover, the norm $\|\cdot\|_\rho^a$ has the equivalent expression

$$\|Z\|_\rho^a = \inf_{k>0} \left\{ \frac{1}{k} (1 + I_\rho(kZ)) \right\}. \quad (3.6)$$

Under these equivalent norms, the linear space L_ρ is a Banach space.

Finally, when ρ is finite the topological dual of E_ρ is isometrically isomorphic to L_{ρ^*} (assuming that in one space a $\|\cdot\|^l$ norm is taken and in the other a $\|\cdot\|^a$ norm is taken) with the identification $[\phi \in E_\rho^* \leftrightarrow g \in L_{\rho^*}] \iff [\phi(Z) = \mathbb{E}(Zg), \forall Z \in E_\rho]$.

The norms $\|\cdot\|_\rho^l$ and $\|\cdot\|_\rho^a$ are called respectively Luxemburg and Amemiya norms. Now thanks to Young's inequality, one can derive a series of Hölder inequalities:

$$\mathbb{E}(|Zg|) \leq 2N_\rho(Z)N_{\rho^*}(g)$$

where N_ρ (resp. N_{ρ^*}) represents any of the norms in L_ρ (resp. L_{ρ^*}) introduced in Theorem 3.4. In particular, L_{ρ^*} (resp. L_ρ) is embedded in the topological dual of L_ρ (resp. L_{ρ^*}), and L_ρ and L_{ρ^*} are continuously embedded in L^1 . The following growth property of a rho-functional and its relation with topological properties of the associated Orlicz-Musielak space is relevant:

Definition 3.5. A finite rho-functional ρ is said to satisfy the Δ_2 condition (or $\rho \in \Delta_2$), if there is a constant $K \geq 1$ and a non-negative integrable function h such that a.s.:

$$\rho(2x, \omega) \leq K\rho(x, \omega) + h(\omega). \quad (3.7)$$

We now state Corollary 1.7.4 in [24] as:

Theorem 3.6. Let ρ satisfy condition Δ_2 . Then $E_\rho = \text{dom}(I_\rho) = L_\rho$ and hence $(L_\rho)^*$ is isometrically isomorphic to L_{ρ^*} . Moreover, if the measure \mathbb{P} is non-atomic, the condition Δ_2 is also necessary for this last isomorphism to hold.

Therefore, if both ρ and ρ^* satisfy the Δ_2 condition, the Banach spaces L_ρ and L_{ρ^*} are in topological duality and are reflexive. The converse is true if \mathbb{P} is non-atomic.

3.2 Towards the robust optimization problem

We next associate a family of Orlicz-Musielak spaces of the previous type with the robust maximization problem (2.4). We recall first some useful and well-known properties of the function V in (2) (see Lemma 2.3.1 in [4]).

Lemma 3.7. The function V is strictly convex, l.s.c. finite and differentiable (on $(0, \infty)$), strictly decreasing, strictly positive, and satisfies:

$$\lim_{x \rightarrow \infty} \frac{V(x)}{x} = \inf\{x : U(x) > -\infty\} \quad \text{and} \quad V(0) = \lim_{x \rightarrow \infty} U(x).$$

Moreover, if U satisfies $AE(U) < 1$, then condition (2.10) holds for V .

The next functions, briefly introduced in Section 2.3, will play a central role in the sequel:

Definition 3.8. For $y \geq 0$ we define the function

$$\gamma_y^*(z) = \begin{cases} \infty & \text{if } z < 0, \\ zV\left(\frac{y}{z}\right) & \text{if } z \geq 0, \end{cases} \quad (3.8)$$

and call γ_y its convex conjugate. We use the convention $\frac{0}{0} = 0$ to define γ_0^* .

In robust optimization (a branch within optimization theory) one would call γ_l^* the *adjoint* of V (see e.g. [7]). The next result is known, except for the third item; see the Appendix for a proof.

Lemma 3.9. Under point 1. in Assumption 2.3, we have

- The function $(y, z) \mapsto \gamma_y^*(z)$ is convex on $[0, \infty)^2$.
- The function $\gamma_y^*(\cdot)$ is l.s.c., convex in its domain (strictly if $y > 0$), on the positive half-line is increasing, finite and strictly positive, and we have $\gamma_y^*(0) = 0$ and $\lim_{t \rightarrow +\infty} \frac{\gamma_y^*(t)}{t} = +\infty$.
- If $y > 0$ then $\gamma_y^*(|\cdot|) = |\cdot|V\left(\frac{y}{|\cdot|}\right)$ and $\gamma_y(|\cdot|) = yU^{-1}(|\cdot|)$ are convex conjugates.

The advantage of working with $\gamma_y^*(|\cdot|)$ is that it is a finite, even function. Notice that we have seen the functions $\gamma_y^*(|\cdot|)$ and $\gamma_y(|\cdot|)$, with $y = Y(\omega) > 0$ in Section 2.1. This motivates

Definition 3.10. Let $Y \in \mathcal{Y}_{\mathbb{P}}(1)$. We denote by $\eta_Y^*, \eta_Y : \mathbb{R} \times \Omega \rightarrow [0, \infty]$ the functionals:

$$\eta_Y^*(z, \omega) := \gamma_{Y_T(\omega)}^*(|z|) = |z|V\left(\frac{Y_T(\omega)}{|z|}\right) \quad \text{and} \quad \eta_Y(z, \omega) := \gamma_{Y_T(\omega)}(|z|) = Y_T(\omega)U^{-1}(|z|).$$

Of course, if $Y_T > 0$ a.s., $\eta_Y^*(\cdot, \omega)$ and $\eta_Y(\cdot, \omega)$ almost surely inherit the obvious properties of $\gamma_y^*(|\cdot|)$ and $\gamma_y(|\cdot|)$ (stated e.g. in Lemma 2.3.3 of [4]). As it is next proved, under mild assumptions they induce rho-functionals:

Proposition 3.11. Let $Y \in \mathcal{Y}_{\mathbb{P}}(1)$ be strictly positive and suppose Assumption 2.3 point 1.

- a) Then the a.s. convex conjugate of the function $\eta_Y^*(\cdot, \omega)$ is $\eta_Y(\cdot, \omega)$ and, provided that

$$\forall \beta > 0, \mathbb{E}[V(\beta Y_T)] < \infty,$$

$\eta_Y^*(\cdot, \omega)$ and $\eta_Y(\cdot, \omega)$ are rho-functionals in the sense of Definition 3.1.

- b) If condition (2.10) (resp (2.11)) holds, the function $\eta_Y^*(\cdot, \omega)$ (resp. $\eta_Y(\cdot, \omega)$) is in Δ_2 .

- c) If $AE(U) < 1$, then $\eta_Y^* \in \Delta_2$ and the condition in a) reduces to

$$\exists \beta > 0, \mathbb{E}[V(\beta Y_T)] < \infty.$$

Proof. The functionals η_Y and η_Y^* are clearly jointly measurable, and the fact that they are conjugate to each other follows from applying Lemma 3.9 almost surely. By properties of U and V , as functions of z they are a.s. l.s.c., even, null at the origin and convergent to 0 at 0 and to infinity at infinity. Also, $\mathbb{E}[Y_T U^{-1}(c)] \leq U^{-1}(c)$ for every constant $c > 0$ since $Y \in \mathcal{Y}_{\mathbb{P}}(1)$ satisfies $\mathbb{E}(Y_T) \leq 1$. Hence, $\eta_Y(c)$ is integrable. The assumption $\mathbb{E}[V(\beta Y_T)] < \infty$ for every $\beta > 0$ implies that also η_Y^* is integrable when applied to

constants. We conclude that they are rho-functionals. For the second point, notice that thanks to (2.10),

$$\begin{aligned}\eta_Y^*(2z) &= 2zV \left(\frac{Y}{2z} \right) \leq 2a\eta_Y^*(z) + 2b(Y+z) \\ &= 2a\eta_Y^*(z) + 2bY + 2bz\mathbf{1}_{\{z \geq Y/V^{-1}(1)\}} + 2bz\mathbf{1}_{\{z < Y/V^{-1}(1)\}} \\ &\leq 2a\eta_Y^*(z) + 2bY + 2b\eta_Y^*(z) + 2bY/V^{-1}(1),\end{aligned}$$

for every $z > 0$, which means that $\eta_Y^* \in \Delta_2$. The corresponding property for η_Y is direct. The last statement c) follows from the last part of Lemma 3.7. \square

Point (c) above should be compared with the comment before Corollary 6.1 in [25]. With some abuse of notation, for $Z \in L^0$ we will write simply $\eta_Y^*(Z)$ referring to the function $\eta_Y^*(Z, \cdot) : \Omega \rightarrow [0, +\infty)$ such that $\eta_Y^*(Z, \cdot)(\omega) = \eta_Y^*(Z(\omega), \omega)$.

Remark 3.12. We deduce that, whenever $Y \in \mathcal{Y}_{\mathbb{P}}(1)$ satisfies $Y_T > 0$ a.s. and $Y \in \mathcal{Y}^*$,

$$L_{\eta_Y^*} = \{Z \in L^0 \text{ s.t. } \exists \alpha > 0, \mathbb{E}^{\mathbb{P}}[\eta_Y^*(\alpha Z)] < \infty\}$$

is an Orlicz-Musielak space. Moreover, $L_{\eta_Y^*}$ and L_{η_Y} (defined analogously) are in separating topological duality and, by Theorem A.5 in [23] or Proposition 1.5 in [24], $\mathbb{E}[\eta_Y^*(\cdot)]$ and $\mathbb{E}[\eta_Y(\cdot)]$ are convex conjugates to each other w.r.t. the given duality.

We end this section commenting that if $Y \in \mathcal{Y}^*$, then the topology of $L_{\eta_Y^*}$ is stronger than that of L^1 and that bounded sets in $L_{\eta_Y^*}$ are uniformly integrable, see Lemma 2.3.5 of [4].

4 Worst-case measures in complete markets

In this section we prove Theorem 2.10. As explained at the outset of Section 2.3, we take the reference measure to be the unique martingale measure, but the result can be generalized if this were not the case, at the price of dealing with random Young functions. Upon introducing the useful notation

$$\eta^*(z) := \eta_1^*(z) = |z|V \left(\frac{1}{|z|} \right) = \gamma_1^*(|z|), \quad z \in \mathbb{R}, \quad (4.1)$$

we recall that the Orlicz-Musielak space pertinent for the problem is the Orlicz space L_{η^*} . Finally, from Lemma 3.9, we know that the conjugate function of η^* is the even function

$$\eta := \bar{\gamma}_1(\cdot) = \gamma_1(|\cdot|) = U^{-1}(|\cdot|).$$

4.1 Characterization of the minimizing measure

In what follows, Assumption 2.9 is enforced. We briefly discuss in details its main consequences, and introduce some additional notation needed in this section. Note first that, under points i) and ii) the integral

$$\int_{\Omega} \langle g, \theta(\omega) \rangle_{\mathbf{G}_0, \mathbf{F}_0} Z(\omega) d\mathbb{P}(\omega),$$

is well defined for each $Z \in L_{\eta^*}$ and all $g \in \mathbf{G}_0$, by Hölder's inequality; it therefore defines the element of $\mathbf{F}_0 = (\mathbf{G}_0)'$ denoted by $\Theta(Z)$ in point iv). We write $\mathbf{F}_1 := \mathbf{F}_0 \times \mathbb{R}$, $\mathbf{G}_1 = \mathbf{G}_0 \times \mathbb{R}$ and $\langle \cdot, \cdot \rangle_{\mathbf{G}_1, \mathbf{F}_1}$ for the obvious duality product between these spaces and set

$$\theta_1(\omega) := (\theta(\omega), 1) \in \mathbf{F}_1, \quad \Theta_1(Z) := \left(\int \theta Z \, d\mathbb{P}, \int Z \, d\mathbb{P} \right) = \int \theta_1 Z \, d\mathbb{P} \in \mathbf{F}_1,$$

and $\mathbf{C}_1 := \mathbf{C}_0 \times \{1\}$. By points i), ii) and iii) the adjoint $\Theta_1^* : \mathbf{G}_1 \rightarrow L_{\eta}$ of $\Theta_1 : L_{\eta^*} \rightarrow \mathbf{F}_1$ given by $\Theta_1^*((g, a))(\omega) = \langle g, \theta(\omega) \rangle_{\mathbf{G}_0, \mathbf{F}_0} + a$ is a linear injection, $g \in \mathbf{G}_1 \mapsto \|\Theta_1^*(g)\|_{\eta}$ defines a norm and \mathbf{G}_1 can be identified with $\Theta_1^*(\mathbf{G}_1)$. Notice that iii) can always be assumed to hold, replacing if needed \mathbf{G}_0 by $\mathbf{G}_0/\mathbf{G}_0^{\mathbb{P}}$, with $\mathbf{G}_0^{\mathbb{P}} := \{g \in \mathbf{G}_0 : \Theta^*(g) = \text{cst. } \mathbb{P} - a.s.\}$, and $\langle \cdot, \cdot \rangle_{\mathbf{G}_0, \mathbf{F}_0}$ by the bi-linear map $(g + \mathbf{G}_0^{\mathbb{P}}, f) \mapsto \langle g, f \rangle_{\mathbf{G}_0, \mathbf{F}_0}$.

The completion \mathbf{G} of \mathbf{G}_1 with respect to $\|\Theta_1^*(\cdot)\|_{\eta}$ is isometrically isomorphic to the closure $\overline{\Theta_1^*(\mathbf{G}_1)}^{L_{\eta}}$ in L_{η} and Θ_1^* has a equally denoted isometric extension to \mathbf{G} . Recall that we write

$$\langle g, \theta_1 \rangle := \Theta_1^*(g),$$

for the element of $\overline{\Theta_1^*(\mathbf{G}_1)}^{L_{\eta}}$ identified with $g \in \mathbf{G}$. The topological dual of \mathbf{G} is

$$\mathbf{F} := \{f \in \mathbf{F}_1 : \exists C_f > 0 \text{ s.t. } |\langle g, f \rangle_{\mathbf{G}_1, \mathbf{F}_1}| \leq C_f \|\Theta_1^*(g)\|_{\eta} \, \forall g \in \mathbf{G}_1\}$$

and we use the notation $\langle \cdot, \cdot \rangle$ for the natural extension of the dual product $\langle \cdot, \cdot \rangle_{\mathbf{G}_1, \mathbf{F}_1}$ to $\mathbf{G} \times \mathbf{F}$. Notice that $\Theta_1 : L_{\eta^*} \mapsto \mathbf{F}$ is continuous and (by Hahn-Banach extension Theorem) surjective; \mathbf{F} can thus be identified with the quotient of L_{η^*} by the annihilator $\left(\overline{\Theta_1^*(\mathbf{G}_1)}^{L_{\eta}}\right)^{\perp}$. One can always choose $Z \in \Theta_1^{-1}(f)$ measurable with respect to the sigma-field generated by $\Theta_1^*(\mathbf{G}_1)$ (replacing Z by $\mathbb{E}(Z|\mathcal{G})$ if needed).

The previous objects being introduced, we can now proceed to the proof of Theorem 2.10. *Part a)* : Observe that functions γ_y^* , γ_y , η_y^* and η_y correspond respectively to functions γ^* , γ , λ_{\diamond}^* and λ_{\diamond} in [28] (with, in the notation therein, $m(z) = 0$ and $\gamma = \lambda$), our mappings θ_1 and Θ_1 correspond respectively to the mappings θ and T_0 therein, and our spaces and sets \mathbf{F}_1 , \mathbf{G}_1 , \mathbf{C}_1 , \mathbf{F} and \mathbf{G} correspond respectively to \mathcal{X}_0 , \mathcal{Y}_0 , C , \mathcal{X} and \mathcal{Y} in that work. Applying parts a) and b) of Theorem 3.2 in [28], and since conditions 1) and 2) therein are ensured by our assumptions, we readily deduce the validity of part a) of Theorem 2.10, except for the attainability of problem (2.15), which requires some additional analysis.

Indeed, note that a solution to (2.15) might in general not exist, since the non-even function γ_y^* does not provide a control of the Young function η^* defining the space L_{η^*} . As in [28], we need to introduce first a suitable extension of (2.15), which will always have a solution in some abstract space under our assumptions, and prove that it actually is an element of \mathbf{G} , which thus solves (2.15). We point out however that the results on the extended dual problem in [28] do not apply here (since our function $w \mapsto \gamma_y((w)_-)$ vanishes) but we will still be able to follow the abstract method of [29] on which [28] relies and conclude similarly.

Let us thus introduce the extension of problem (2.15). We denote by \widetilde{L}_{η} the algebraic dual of L_{η^*} and by $\langle \cdot, \cdot \rangle$ the associated dual product. We also consider the space $\widetilde{\mathbf{G}}$ defined as the algebraic dual of \mathbf{F} , and we write $\langle \cdot, \cdot \rangle$ for the corresponding dual product as well (which dual product is meant should be clear from the context). Observe that the operator $\Theta_1 : L_{\eta^*} \rightarrow \mathbf{F}$ naturally induces the extension $\Theta_1^* : \widetilde{\mathbf{G}} \rightarrow \widetilde{L}_{\eta}$ of $\Theta_1^* : \mathbf{G} \rightarrow L_{\eta}$ given by

$$\langle \Theta_1^*(g), Z \rangle = \langle g, \Theta_1(Z) \rangle, \quad (g, Z) \in \widetilde{\mathbf{G}} \times L_{\eta^*}.$$

Introduce also the convex functions $\Phi_y(W) := y \int \gamma(W) d\mathbb{P}$, $W \in L_\eta$,

$$\Phi_y^*(Z) := \int \gamma_y^*(Z) d\mathbb{P} = \sup_{W \in L_\eta} \mathbb{E}(ZW) - y \int \gamma(W) d\mathbb{P}, \quad Z \in L_{\eta^*} \quad (4.2)$$

the last equality, thanks to Proposition 3.11 a), and

$$\bar{\Phi}_y(\zeta) := \sup_{Z \in L_{\eta^*}} \langle \zeta, Z \rangle - \Phi_y^*(Z), \quad \zeta \in \widetilde{L}_\eta.$$

With this elements, the extended dual problem is defined as:

$$\text{Maximize } \inf_{f \in \mathbf{C}} \langle g, f \rangle - \bar{\Phi}_y(\Theta_1^*(g)), \quad g \in \widetilde{\mathbf{G}}. \quad (\widetilde{D}_y)$$

Recall next that a topological vector space L endowed with a partial order \leq is called a Riesz space if \leq is a lattice: $\forall \ell_1, \ell_2 \in L, \exists \ell_1 \vee \ell_2 \in L$ such that $\ell_1 \vee \ell_2 \geq \ell_1, \ell_2$ and $\ell_1 \vee \ell_2 \leq \ell$ $\forall \ell \in L$ such that $\ell \geq \ell_i, i = 1, 2$. Given $\ell \in L$, the elements ℓ_+, ℓ_- and $|\ell|$ are then defined in a similar way as in \mathbb{R} . A dual order also written \leq is induced in the algebraic dual L' of L . By Riesz' Theorem, the space $L^b := \{\zeta \in L' : \sup_{\ell' \in L, |\ell'| \leq \ell} |\langle \zeta, \ell' \rangle| < \infty \forall \ell \in L, \ell \geq 0\}$ of

“relatively bounded linear forms”, or order dual of L , is a Riesz space too. In particular, $\zeta \in L^b$ admits a unique decomposition $\zeta = \zeta_+ - \zeta_-$ into positive and negative parts $\zeta_+, \zeta_- = (-\zeta)_+ \in L^b$, with $\zeta_+ \geq 0$ and $\langle \zeta_+, \ell \rangle := \sup_{\ell' \in L, 0 \leq \ell' \leq \ell} \langle \zeta, \ell' \rangle$. A complete Riesz space

L endowed with a norm $\|\cdot\|$ such that $|\ell_1| \leq |\ell_2| \Rightarrow \|\ell_1\| \leq \|\ell_2\|$ is called a Banach lattice, and its order dual L^b and topological dual L^* coincide. We refer the reader to [1] Ch. 8 and 9 for these facts and background on Riesz spaces.

The remainder statement in part a) of Theorem 2.10, i.e. the existence of a solution to (2.15) will follow from the two next results:

Lemma 4.1. *Suppose Assumption 2.9 holds and that $\mathbf{C}_1 \cap \text{icor dom}(\Gamma_y^*) \neq \emptyset$. Then, the extended dual problem \widetilde{D}_y has a solution.*

Proof. Existence follows applying Theorem 5.3 in [28] to $\mathcal{U} = L_\eta = \mathcal{U}''$, $\mathcal{L} = L_{\eta^*}$, $\mathcal{X} = \mathbf{F}$ and $\mathcal{Y} = \mathbf{G}$ with our functions Φ_y and Θ_1 in the respective roles of functions Φ_0 and T_0 therein (notice that we have interchanged here the roles of the symbols ' and *, used therein to respectively denote topological or algebraic dual spaces). \square

Lemma 4.2. *Let $\zeta \in \widetilde{L}_\eta$ be such that $\bar{\Phi}_y(\zeta) < \infty$. Then, ζ belongs to the order dual of $Z \in L_{\eta^*}$; in particular there exists $W^\zeta \in L_\eta$ such that $\langle \zeta, Z \rangle = \mathbb{E}(W^\zeta Z)$ for all $Z \in L_{\eta^*}$. Moreover, we have*

$$\bar{\Phi}_y(\zeta) = \bar{\Phi}_y(\zeta_+) = \bar{\Phi}_{y,+}(\zeta) = \Phi_y((W^\zeta)_+) \quad (4.3)$$

where for $\zeta \in \widetilde{L}_\eta$ we define $\bar{\Phi}_{y,+}(\zeta) := \sup_{Z \in L_{\eta^*}} \langle \zeta, Z \rangle - \Phi_{y,+}^*(Z)$, with $\Phi_{y,+}^*(Z) := \int \gamma_y^*(|Z|) d\mathbb{P}$.

The proof of Lemma 4.2 relies on Proposition 5.10 in [28] and is given in the Appendix.

We can now finish the proof of part a) of Theorem 2.10. Indeed, Lemma 4.1 ensures the existence of a solution $\tilde{g} \in \widetilde{\mathbf{G}}$ to \widetilde{D}_y which, thanks to Lemma 4.2, is such that $\Theta_1^*(\tilde{g}) \in L_\eta$. By Theorem 5.7, a) in [28] (taking there $\mathcal{X} = \mathbf{F}$, $\mathcal{X}^* = \widetilde{\mathbf{G}}$ and $\Lambda := \bar{\Phi}_y \circ \Theta_1^* : \mathbf{G} \rightarrow [0, \infty]$), for some net $\{g_\alpha\} \subset \mathbf{G}$ such that $\bar{\Phi}_y(\Theta_1^*(g_\alpha)) < \infty$ one has $\langle g_\alpha, \Theta_1(Z) \rangle \rightarrow \langle \tilde{g}, \Theta_1(Z) \rangle$ for all $Z \in L_{\eta^*}$. In other words, $\Theta_1^*(g_\alpha) \in L_\eta$ converges $\sigma(L_\eta, L_{\eta^*})$ -weakly to $\Theta_1^*(\tilde{g}) \in L_\eta$.

The set $\Theta_1^*(\mathbf{G}_1)$ being convex, its $\sigma(L_\eta, L_{\eta^*})$ -weak closure and its norm closure in L_η coincide. The previous and the definition of \mathbf{G} thus imply that $\tilde{g} \in \mathbf{G}$, hence \tilde{g} solves problem (2.15) too.

Part b): We use Theorem 5.4 in [28] stating that, in the present context, a pair $(Z, g) \in L_{\eta^*} \times \tilde{\mathbf{G}}$ is such that Z solves (2.14) and g solves \tilde{D}_y if and only if the following hold:

$$\begin{cases} \bullet & \Theta_1(Z) \in \mathbf{C} \\ \bullet & \langle \Theta_1^*(g), Z \rangle_{\widetilde{L}_\eta, L_{\eta^*}} \leq \langle \Theta_1^*(g), Z' \rangle_{\widetilde{L}_\eta, L_{\eta^*}} \text{ for all } Z' \in \text{dom } \Phi_y^* \text{ such that } \Theta_1(Z') \in \mathbf{C} \\ \bullet & Z \in \partial_{L_{\eta^*}} \overline{\Phi_y}(\Theta_1^*(g)). \end{cases} \quad (4.4)$$

Note that, since $\gamma(-|\cdot|) = 0$, by part c) of Proposition 5.10 in [28] the third point is always equivalent to $Z \geq 0$ and $Z \in \partial_{L_{\eta^*}} \overline{\Phi_y}(\Theta_1^*(g))$.

Now, assume that $(Z, g) \in L_{\eta^*} \times \mathbf{G}$ solve (2.14) and (2.15). We have from (4.3) that $Z \in \partial_{L_{\eta^*}} \Phi_y((\langle g, \theta_1 \rangle)_+)$. Observe that Φ_y is Gâteaux differentiable in L_η with derivative at point $W \in L_\eta$ given by $y\gamma'(W) \in L_{\eta^*}$, as follows by dominated convergence using the equality $\gamma(W) + \gamma^* \circ \gamma'(W) = W\gamma'(W)$ and the bounds $\gamma(2z) - \gamma(z) \geq \gamma'(z)z$ if $z \geq 0$ (by mean value theorem and increasingness of γ) and $(k-1)\gamma(z) + d \geq \gamma'(z)z$ (with the notation in (2.11) and where, necessarily, $k \geq 1$). We deduce that the third point in b) of Theorem 2.10 is satisfied. Moreover, the space \widetilde{L}_η in the second point in (4.4) can be replaced by L_η . Using the “little dual equality” deduced from part a) when $C_0 = \{f\}$ is a singleton with $f \in \mathbf{F}$:

$$\Gamma_y^*(f) = \{\Phi_y^*(Z) : Z \in L_{\eta^*}, \Theta_1(Z) = f\} (= \{\mathbb{E}(\gamma_y^*(Z)) : Z \in L_{\eta^*}, \Theta_1(Z) = f\}) \quad (4.5)$$

(or proved in part a) of Proposition 5.7 of [29]), we easily obtain, with the surjectivity of Θ_1 the first and second points in part b) of Theorem 2.10.

Reciprocally, if the pair $(Z, g) \in L_{\eta^*} \times \mathbf{G} \subset L_{\eta^*} \times \tilde{\mathbf{G}}$ satisfies the three points in the statement, in a similar way it is seen to satisfy the conditions in (4.4) and thus solve (2.14) and \tilde{D}_y . Since $g \in \mathbf{G}$, it solves (2.15) and the proof of part b) of Theorem 2.10 is finished.

Part c): since the assumptions of Theorem 2.4 hold true under Assumption 2.9, part c) of Theorem 2.10 follows from part a) since u and v are conjugate.

5 Modular spaces and the incomplete case

In this section, the robust optimization problem in the incomplete market case will be explored. Essentially the aim is to prove here Theorems 2.4 and 2.5. In Subsection 5.1 the natural extension from the Orlicz-Musielak setting to the modular one will be motivated. Likewise the potential usefulness of this extension to the robust optimization problem will be sketched. Then in Subsection 5.2 and the following one, the machinery of modular spaces and its link to the problem of robust optimization will be fully explored. The main result here is the proof of Theorem 2.4. The second crucial result is then Theorem 5.14, a slight extension of Theorem 2.5, and the remarks thereafter.

5.1 Modular space associated with the incomplete case

Let us recall the notation :

$$\eta_Y^*(z) = |z|V(Y/|z|) \quad \text{and} \quad \eta_Y(x) = YU^{-1}(|x|),$$

of Definition 3.10 and re-introduce the important functionals we already saw in Section 2.1:

$$\begin{aligned} I(Z) &:= \inf_{Y \in \mathcal{Y}} \mathbb{E}[\eta_Y^*(Z)] = \inf_{Y \in \mathcal{Y}} \mathbb{E}[|Z|V(Y/|Z|)], \\ J(X) &:= \sup_{Y \in \mathcal{Y}} \mathbb{E}[\eta_Y(X)] = \sup_{Y \in \mathcal{Y}} \mathbb{E}[YU^{-1}(|X|)]. \end{aligned}$$

We start by observing that for the set

$$\mathcal{Y}^* = \{Y \in \mathcal{Y} : Y > 0 \text{ a.s. and } \forall \beta > 0, \mathbb{E}[V(\beta Y)] < \infty\},$$

and under Assumption 2.3, we may compute I and J on \mathcal{Y}^* simply. More exactly:

Lemma 5.1. *We have:*

(i) *If point 2. in Assumption 2.3 holds, then*

$$I(Z) = \inf_{Y \in \mathcal{Y}^*} \mathbb{E}[\eta_Y^*(Z)] \quad \text{and} \quad J(X) = \sup_{Y \in \mathcal{Y}^*} \mathbb{E}[\eta_Y(X)].$$

(ii) *If either the reference measure \mathbb{P} is a martingale measure, or there is a continuous \mathbb{P} -local martingale M and $\lambda \in L^2(M)$ such that the price process satisfies $dS_t = dM_t + \lambda_t \cdot d\langle M \rangle_t$ and $\mathbb{E}[V(\beta \mathcal{E}(-\int \lambda dM)_T)] < \infty$ for every $\beta > 0$, where \mathcal{E} stands for the stochastic exponential, then $\mathcal{Y}^* \neq \emptyset$, i.e. point 2. in Assumption 2.3 holds.*

From the previous lemma, we can always assume to be working with $Y \in \mathcal{Y}^*$ at will. The proof is given in the appendix.

Recall that $\eta_Y^*(z) = |z|V(Y/|z|)$ is a “random Young function” induced by $Y \in \mathcal{Y}$. For $Y \in \mathcal{Y}^*$ such functions induce a space $L_{\eta_Y^*} = \{Z \in L^0 : \mathbb{E}[\eta_Y^*(\alpha Z)] < \infty, \text{ some } \alpha > 0\}$, called Orlicz-Musielak space (see Proposition 3.11), which we will denote here L_Y^* for simplicity. These spaces have, as discussed in Theorem 3.4, several equivalent norms; for instance the Luxemburg or the Amemiya norms, respectively:

$$\|Z\|_Y^l := \inf\{\beta > 0 : \mathbb{E}[\eta_Y^*(\beta Z)] \leq 1\} \quad \text{and} \quad \|Z\|_Y^a := \inf_{k>0} \left[\frac{1}{k} + \frac{\mathbb{E}[\eta_Y^*(kZ)]}{k} \right].$$

We also define the spaces L_Y analogously, in terms of η_Y , the conjugate of η_Y^* .

It is then clear that $v(y) = y \inf_{Z \in d\mathcal{Q}_e/d\mathbb{P}} I(Z/y)$. On the other hand, recall that the function $(Y, Z) \in (L^0)_+ \times (L^0)_+ \mapsto \mathbb{E}[ZV(Y/Z)]$ is jointly convex (as $(y, z) \rightarrow zV(y/z)$ is so) and jointly lower-semicontinuous w.r.t. convergence in probability (see the proof of Lemma 3.7 in [39]). Also recall the following Komlos-type argument (see Lemma A.1.1 in [14]): if $\{A_n\}_n$ is a sequence of positive random variables bounded in L^0 , then there is a positive finite r.v. A and a sequence $B_n \in \text{conv}\{A_n, A_{n+1}, \dots\}$ such that $B_n \rightarrow A$ in probability.

We associate to the functional I a set, in complete analogy to Orlicz-Musielak spaces:

$$L_I := \{Z \in L^0(\mathbb{P}) : I(\alpha Z) < \infty \text{ for some } \alpha > 0\}, \quad (5.1)$$

and define L_J accordingly in terms of J . Now we collect some elementary observations. The reader should notice that these spaces coincide with the ones given in Section 2.1.

Lemma 5.2. *The following hold:*

- *The functionals $I, J : (L^0)_+ \rightarrow [0, \infty]$ are convex and moreover I is lower-semicontinuous w.r.t. convergence in measure. Also, for each non-vanishing $Z \in \text{dom}(I)$, the infimum in $I(Z)$ is attained at some $Y \in \mathcal{Y}$.*

- The set L_I is a linear space coinciding with $\cup_{Y \in \mathcal{Y}} L_Y^*$, whereas the set L_J is a linear space contained in $\cap_{Y \in \mathcal{Y}} L_Y$.
- $J(M) \leq x \iff U^{-1}(|M|) \leq X_T$ for some $X \in \mathcal{X}(x)$.

Proof. For the convexity of I , recall that the partial infimum of every jointly convex function is convex. The fact that $I(Z)$ is attained is a consequence of the closedness and convexity of \mathcal{Y} , a Komlos-type argument and the lower semicontinuity of $Y \mapsto \mathbb{E}[ZV(Y/Z)]$. This in turn implies the lower semicontinuity of I , now because $(Y, Z) \mapsto \mathbb{E}[ZV(Y/Z)]$ is l.s.c. That J is convex is a consequence of the convexity of U^{-1} . The equality of the sets mentioned in the second point is evident from the fact that for Z fixed the infimum over the $Y \in \mathcal{Y}$ is attained. The linearity of L_I follows now from the convexity of I : if $I(\alpha Z), I(\beta X) < \infty$, taking $\gamma = \frac{\alpha\beta}{\alpha+\beta}$ yields $I(\gamma[Z + X]) = I\left[\frac{\beta}{\alpha+\beta}[\alpha Z] + \frac{\alpha}{\alpha+\beta}[\beta X]\right] \leq \frac{\beta}{\alpha+\beta}I(\alpha Z) + \frac{\alpha}{\alpha+\beta}I(\beta X) < \infty$. The linearity of L_J is proved as in the case of L_I . It is clear that if $X \in L_J$ then also $X \in L_Y$, for every $Y \in \mathcal{Y}$. The last point goes by definition of J and Proposition 3.1.ii in [25]. \square

We shall see in the next section that $|Z|_I^q = \inf_{k>0} [\frac{1}{k} + \frac{I(kZ)}{k}]$ is a norm on L_I , making it a Banach space. Further this norm-topology will be stronger than that of convergence in measure. This implies immediately that I will be lower-semicontinuous with respect to $|\cdot|_I^q$. In light of this, let us justify the appeal of the space L_I :

Remark 5.3. Since $v(y) = y \inf_{Z \in d\mathcal{Q}_e/d\mathbb{P}} I(Z/y)$, and also by definition $|Z|_I^q \leq y + yI(Z/y)$, by taking a minimizing sequence $\{Z_n\}$ such that $yI(Z_n/y)$ decreases to $v(y)$ it follows that the sequence $\{Z_n\}$ would be bounded in $(L_I, |\cdot|_I^q)$. On the other hand, we shall see in Proposition 5.11 that $u_{\mathbb{Q}}(x) \geq c|Z|_I^q$. This shows that in minimizing the $u_{\mathbb{Q}}$'s we may restrict \mathcal{Q} to its intersection with a given ball. Hence the two previous estimates show that requiring $d\mathcal{Q}/d\mathbb{P}$ to be closed in $(L_I, |\cdot|_I^q)$ and asking for conditions on the ingredients of the problem so that this space becomes reflexive, would allow to fully solve the robust optimization problem. We will see, however, that L_I is reflexive *almost exactly* when the market is complete, and that this is independent of how well-behaved our utility function is (in stark contrast to the complete case). On the other hand, because we will be able to prove that L_J is a norm-dual space, and since the image through U of the terminal wealths live in this space and are uniformly norm-bounded, we can still derive a minimax identity.

5.2 Modular spaces L_F and E_F ; topological/duality results

Generating a space from a functional is a classical subject. See e.g. [31, 32]. There are quite minimalistic conditions ensuring that the generated space be an F -space and that some related functionals form a family of pseudo-norms for it. Here, rather than working at this level of generality, a more relaxed terminology and a lighter approach (as in chapter XI in [32]) will be pursued.

We first introduce the notion of a convex modular, and then its associated modular space. We shall see that I (respect. J) and L_I (respect. L_J) fulfil these definitions.

Definition 5.4. A functional $F : \mathcal{S} \rightarrow [0, \infty]$ over a vector space \mathcal{S} is called a *Convex modular* if the following axioms are fulfilled:

1. $F(0) = 0$
2. $F(s) = F(-s)$

3. $\forall s \in \mathcal{S}, \exists \lambda > 0 : F(\lambda s) < \infty$
4. $F(\xi s) = 0$ for every $\xi > 0$ implies $s = 0$
5. F is convex
6. $F(s) = \sup_{0 \leq \xi < 1} F(\xi s)$

With this definition, it follows that on the space:

$$L_F(\mathcal{S}) := \{s \in \mathcal{S} : \lim_{\alpha \rightarrow 0} F(\alpha s) = 0\} = \{s \in \mathcal{S} : F(\alpha s) < \infty \text{ for some } \alpha > 0\}$$

the following functionals are equivalent norms, called respectively Luxemburg and Amemiya norms:

$$|s|_F^l = \inf\{\beta > 0 : F(s/\beta) \leq 1\} \text{ and } |s|_F^a = \inf\left\{\frac{1}{k} + \frac{F(ks)}{k} : k > 0\right\},$$

and actually thanks to Theorem 1.10 in [31], $|s|_F^l \leq |s|_F^a \leq 2|s|_F^l$. It can be proved, as in chapter XI, 81 in [32], that the topology induced by the Luxemburg norm is exactly the (weakest locally convex topology) generated by the family of neighbourhoods of the origin $\{F^{-1}(-\infty, c]\}_c$. The space L_F is called a *modular space associated to F* .

Now recalling the definitions in the previous subsection, we prove:

Proposition 5.5. *The functional I is a convex modular and L_I is a modular space associated to it. Likewise, J is a convex modular and L_J is a modular space associated to it.*

Proof. For I first. Axioms (1), (2) and (3) hold by definition, and (5) is proved in Lemma 5.2. For (4) notice that $I(\xi Z) = 0$ implies $\mathbb{E}[ZV(Y/(\xi Z))] = 0$ for some $Y \in \mathcal{Y}$. By positivity, this shows $ZV(Y/(\xi Z)) = 0$ a.s., from where $Z = 0$ a.s. Finally, for axiom (6), first recall that $z \mapsto zV(Y/z)$ is increasing, from which $I(Z) \geq \sup_{0 \leq \xi < 1} I(\xi Z) =: \zeta$. Now, take $\epsilon_n \nearrow 1$ so $\zeta = \lim I(\epsilon_n Z)$. Because I is l.s.c. we deduce that $\lim I(\epsilon_n Z) \geq I(Z)$ and thus $I(Z) = \zeta$.

Now for J . Axioms (1), (2) and (3) are direct. If $J(\xi X) = 0$ this means $YU^{-1}(\xi X) = 0$, for all $Y \in \mathcal{Y}$ a.s. Thus $X = 0$ a.s. Lastly, by increasingness of U^{-1} it holds that for fixed Y : $YU^{-1}(\xi X) \nearrow YU^{-1}(X)$ as $\xi \nearrow 1$. By monotone convergence then $\mathbb{E}[YU^{-1}(\xi X)] \nearrow \mathbb{E}[YU^{-1}(X)]$ and thus $\sup_{0 \leq \xi < 1} \mathbb{E}[YU^{-1}(\xi X)] = \mathbb{E}[YU^{-1}(X)]$ and now taking supremum over $Y \in \mathcal{Y}$ we get axiom (6). □

Call now L_I^* and L_J^* the topological duals. By the “reflexivity Theorem” in [33] it holds automatically that the modulars J and I are reflexive, in the sense that if the following functionals are defined:

$$I^*(l) := \sup_{Z \in L_I} \{l(Z) - I(Z)\} \text{ for } l \in L_I^* \quad \text{and} \quad J^*(j) := \sup_{X \in L_J} \{j(X) - J(X)\} \text{ for } j \in L_J^*,$$

then I and J may be recovered, that is:

$$I(Z) = \sup_{l \in L_I^*} \{l(Z) - I^*(l)\} \quad \text{and} \quad J(X) = \sup_{j \in L_J^*} \{j(X) - J^*(j)\}.$$

In particular then, both I and J are lower semicontinuous under the strong topologies introduced thus far, and by convexity, also under their weak topologies. What is more,

from Lemma 5.6, part 1), we deduce by Theorem 5.43 in [1] that both functionals are norm-continuous in the interior of their domains.

Another space of interest is the so-called set of finite elements of a modular space L_F , denoted E_F , which typically has better properties:

$$E_F = \{s \in \mathcal{S} : F(\alpha s) < \infty \text{ for all } \alpha > 0\}.$$

We remark that $E_I = L_I = \text{dom}(I)$ as soon as condition (2.10) in Assumption 2.2 holds. Let us state now a few results that will be repeatedly useful:

Lemma 5.6. *For every $Z \in L_I$, $X \in L_J$:*

1. $I\left(\frac{Z}{|Z|_I^l}\right) \leq 1$ and $J\left(\frac{X}{|X|_J^l}\right) \leq 1$.
2. Z_n norm converges to Z in L_I (respect. X_n norm converges to X in L_J) if and only if for all $\alpha > 0$, $I(\alpha[Z_n - Z]) \rightarrow 0$ (respect. $J(\alpha[X_n - X]) \rightarrow 0$).
3. $I(Z) + J(X) \geq \mathbb{E}[XZ]$.

Proof. We prove (1) first. Notice $J\left(\frac{X}{|X|_J^l}\right) \leq \sup_Y \mathbb{E}[YU^{-1}(X/\|X\|_{\eta_Y}^l)] \leq 1$, the first inequality because clearly $|X|_J^l \geq \|X\|_{\eta_Y}^l$ and the second by definition of the Luxemburg norm and Fatou's Lemma. On the other hand take $\beta_n \searrow |Z|_I^l$ such that $I(Z/\beta_n) \leq 1$. Since $Z/\beta_n \rightarrow Z/|Z|_I^l$ in probability we conclude by Lemma 5.2 that

$$I(Z/|Z|_I^l) \leq \liminf I(Z/\beta_n) \leq 1.$$

Part (2) is a direct consequence of Theorem 3 in Chapter XI,81 of [32]. For part (3), by Remark 3.12 the conjugate of η_Y is η_Y^* , and so $\mathbb{E}[XZ] \leq \mathbb{E}[ZV(Y/Z)] + \mathbb{E}[YU^{-1}(X)]$ for every $Y \in \mathcal{Y}^*$. Thus bounding $\mathbb{E}[YU^{-1}(X)]$ above by $J(X)$ and then taking infimum over $Y \in \mathcal{Y}$ yields $\mathbb{E}[XZ] \leq I(Z) + J(X)$. □

Time is ripe to prove some more refined properties of the spaces L_I and L_J . Fortunately Lemma 5.1 says that the properties of both L_Y and L_Y^* , with $Y \in \mathcal{Y}^*$, can be lifted.

Proposition 5.7. *Both subspaces E_I and E_J are closed subspaces of L_I and L_J respectively. When considering the almost-sure ordering, E_I and L_J are Banach lattices, and furthermore E_I is order-continuous.*

In the last result, any of the previously defined norms may have been used. See the Appendix for the lengthy proof.

In order to further understand the modular spaces introduced thus far, and in doing so paving the way for the central statements of this section, some duality results will be pursued. First of all, Hölder-type inequalities are proved:

Proposition 5.8. *We have:*

$$|\mathbb{E}[XZ]| \leq |Z|_I^i |X|_J^j \leq 2|Z|_I^k |X|_J^k,$$

where $i, j, k \in \{a, l\}$ and $i \neq j$. Furthermore, the inclusions $L^\infty \rightarrow L_J \rightarrow L^1$ and $L^\infty \rightarrow L_I \rightarrow L^1$ are continuous.

Proof. From inequality (3) in Lemma 5.6 follows that $\mathbb{E}[XZ] \leq \frac{1}{\alpha\beta}\{I(\alpha Z) + J(\beta X)\}$. Now, take β such that $J(\beta X) \leq 1$. Then $\mathbb{E}[XZ] \leq \frac{1}{\beta} \left[\frac{1}{\alpha}\{1 + I(\alpha Z)\} \right]$ and taking infimum over $\alpha > 0$ yields $\mathbb{E}[XZ] \leq \frac{1}{\beta}|Z|_I^a$. Now taking infimum of the $1/\beta$ such that $J(\beta X) \leq 1$ gives $\mathbb{E}[XZ] \leq |X|_J^l |Z|_I^a$. From here also $|\mathbb{E}[XZ]| \leq |X|_J^l |Z|_I^a$ and by a similar argument $|\mathbb{E}[XZ]| \leq |X|_J^a |Z|_I^l$. Finally, because in the general context of modular spaces (see [32], Chapter XI) holds that $|\cdot|^l \leq |\cdot|^a \leq 2|\cdot|^l$ we get the desired inequalities. Evidently $1 \in L_J$ and by Assumption 2.3 also $1 \in L_I$. By using the derived Hölder inequalities, this shows the continuity of the inclusions into L^1 . On the other hand, because both I and J are increasing, $|\cdot|_I \leq |\cdot|_\infty 1_I$ and likewise for J , thus proving the continuity of the inclusions from L^∞ . \square

Notice from this that, as it can be expected, for every $X \in L_J$ the functional $l_X(\cdot) = \mathbb{E}[\cdot X]$ belongs to L_I^* and for every $Z \in L_I$ the functional $l_Z(\cdot) = \mathbb{E}[\cdot Z]$ belongs to L_J^* . We state now a Riesz-type representation result. This will rest in a few technical points to be established in Lemma 5.10. Both proofs are given in the Appendix.

Proposition 5.9. *The topological dual of E_I is L_J , with the usual identification:*

$$l \in (E_I)^* \leftrightarrow l(Z) = \mathbb{E}[ZX] \text{ for some } X \in L_J,$$

and this identification is isomorphic isometric between $(E_I, |\cdot|_I^a)$ and $(L_J, |\cdot|_J^l)$. Furthermore, for every $Z \in L_I, X \in L_J$, we have $I^(l_X) = J(X)$, and if $E_I = L_I$ also $J^*(l_Z) = I(Z)$.*

Lemma 5.10.

1. $\mathbb{1}_A \in E_I$ for every $A \in \mathcal{F}$
2. Simple functions are norm dense in E_I
3. If $Z_n \rightarrow 0$ a.s. and $|Z_n|$ is bounded by a constant, then $|Z_n|_I \rightarrow 0$
4. If $\kappa := \sup\{|\mathbb{E}[fg]| : f \text{ simple and } |f|_I^a \leq 1\} < \infty$ then $g \in L_J$ and $|g|_J^l = \kappa$

Notice that a property analogous to point (3) in the above lemma does not hold in E_J if \mathcal{Y} is not uniformly integrable.

5.3 Applications of the modular approach to the robust optimization problem

As a consequence of Proposition 5.8, we can prove the following result, of interest on its own, which we already mentioned in Remark 5.3 and will be useful in proving the general minimax Theorem 2.4 below:

Proposition 5.11. *Under Assumption 2.3, for all $x > 0$ we have that*

$$\forall \mathbb{Q} \in \mathcal{Q}: \quad (1+x) \left| \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_I^l \geq u_{\mathbb{Q}}(x) \geq (1 \wedge x) \left| \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_I^a. \quad (5.2)$$

Proof. By Proposition 5.8 we have:

$$\mathbb{E}^{\mathbb{Q}}[U(X_T)] \leq |d\mathbb{Q}/d\mathbb{P}|_I^l |U(X_T)|_J^a \leq [1 + J(U(X_T))] |d\mathbb{Q}/d\mathbb{P}|_I^l,$$

by definition of the norm. Hence, by Lemma 5.2 we get that $u_{\mathbb{Q}}(x) \leq [1+x]|d\mathbb{Q}/d\mathbb{P}|_I^l$. Now we prove the lower bound for $u_{\mathbb{Q}}(x)$ in (5.2). Let us call $Z = \frac{d\mathbb{Q}}{d\mathbb{P}} \in \frac{d\mathcal{Q}_e}{d\mathbb{P}}$. Recalling that $v_{\mathbb{Q}}(y) := \inf_{Y \in \mathcal{Y}} \mathbb{E}[ZV(yY/Z)]$, we have:

$$|Z|_I^a \leq y + yI(Z/y) = y + v_{\mathbb{Q}}(y) \leq y + cv_{\mathbb{Q}}(y),$$

for each $c \geq 1$. Calling $A_{\mathbb{Q}}(y) = v_{\mathbb{Q}}(y) + xy$, then $A_{\mathbb{Q}}(y) \geq \frac{1}{c}|Z|_I + (x - \frac{1}{c})y$. Thus for every $x > 0$, finding $c \geq 1$ such that $x \geq c^{-1}$ and then taking infimum over $\{y > 0\}$ yields $u_{\mathbb{Q}}(x) \geq C|Z|_I$: if the r.h.s. is infinite there is nothing to prove, and otherwise by Theorem 3.1 in [25] it holds $u_{\mathbb{Q}}(x) = \inf_{y>0} [v_{\mathbb{Q}}(y) + xy]$ and we still get the desired bound. The best constant C is thus $1 \wedge x$.

If now $Z := d\mathbb{Q}/d\mathbb{P} \in \frac{d\mathcal{Q}}{d\mathbb{P}} \setminus \frac{d\mathcal{Q}_e}{d\mathbb{P}}$, an easy application of Lemma 3.3 in [39] allows to conclude, from the previous bounds. \square

Thanks to Proposition 5.9 we can endow L_J with a decent weak-* topology and thus finally prove one of our main results for incomplete markets: Theorem 2.4.

Proof. of Theorem 2.4 Fix $x > 0$. We intend to apply Theorem 7, chapter 6, in [2] (Lopsided minimax Theorem, also stated on page 295 therein). First, let us define the set $G := \{g \in L_J : 0 \leq g \leq U(X_T), \text{ some } X \in \mathcal{X}(x)\}$. Now we define a bilinear function $F : G \times d\mathcal{Q}/d\mathbb{P} \rightarrow [0, \infty)$ by $F(g, Z) = \mathbb{E}[Zg]$. Evidently under condition $L_I^* \cong L_J$ we must have that $E_I = L_I$ (which is the case anyway if condition (2.10) in Assumption 2.2 holds).

We first endow the convex set G with the weak-* topology $\sigma(L_J, E_I)$. Let us prove that G is closed with it. Indeed if $\{g_{\alpha}\}_{\alpha} \subset G$, we have by Lemma 5.2, part c), that $J(g_{\alpha}) \leq x$. But by Proposition 5.9, the spaces $(E_I, \sigma(E_I, L_J)), (L_J, \sigma(L_J, E_I))$ are in topological duality and $J = I^*$. Therefore J is $\sigma(L_J, E_I)$ -l.s.c. and we conclude that if $g_{\alpha} \rightarrow g$ in this topology, then $J(g) \leq x$. Again by Lemma 5.2, part c), we see that $|g| \in G$. On the other hand $\mathbf{1}_{g<0} \in E_I$ (by Lemma 5.10) and so $\mathbb{E}[g\mathbf{1}_{g<0}] = \lim \mathbb{E}[g_{\alpha}\mathbf{1}_{g<0}] \geq 0$, from which $g \geq 0$ and so $g \in G$.

We now prove that G is weak*-compact. By Banach-Alaoglu it suffices to prove that it is norm bounded. But this holds since $|g|_J^a \leq 1 + J(g) \leq 1 + x$, for every $g \in G$.

We apply now the lopsided minimax Theorem. The function F satisfies:

- $F(g, \cdot)$ is convex
- $\{g \in G : F(g, Z) \geq \beta\}$ is weak*-compact for every β, Z .
- $F(\cdot, Z)$ is concave and continuous,

and thus $-F$ satisfies with ease the requirements of that theorem. We conclude then the minimax equality and the attainability of an optimal $g \in G$. By simple arguments in [39] (see the proof of Lemma 3.4 therein) any optimal g must be of the form $U(X_T)$ and one may approximate the *infsup* by taking the infimum over \mathcal{Q}_e .

Because we proved that $u(x) = \inf_{\mathbb{Q} \in \mathcal{Q}_e} u_{\mathbb{Q}}(x)$ we also have $u(x) = \inf_{\mathbb{Q} \in \mathcal{Q}_e, u_{\mathbb{Q}}(x) < \infty} u_{\mathbb{Q}}(x)$. Now applying Theorem 3.1 in [25] we see that $u(x) = \inf_{y \geq 0} [\inf_{\mathbb{Q} \in \mathcal{Q}_e, u_{\mathbb{Q}}(x) < \infty} v_{\mathbb{Q}}(y) + xy]$ and so by the first statement in Lemma 3.5 in [39] we conclude that u is the conjugate of v . Finiteness of v on $(0, \infty)$ is a consequence of $L_I = E_I$. Because I is convex and $v(y) = \inf_{Z \in d\mathcal{Q}_e/d\mathbb{P}} yI(Z/y)$, an argument as in the proof of Lemma 3.9 shows that v is convex and so we conclude by Theorem 7.22 in [1] that v is continuous in $(0, \infty)$. Since clearly $v(y) \geq V(y)$ we see that $v(0+) = \infty$. Thus defining $v(\cdot) = \infty$ on $(-\infty, 0]$ we get a l.s.c. function everywhere. Defining $u(0) = 0$ and $u(x) = -\infty$ if $x < 0$, we still get that u is the concave conjugate of v . This in turn implies that v is conjugate to u and also that

if $y > 0$ then $v(y) = \sup_{x>0} [u(x) + xy]$.

Finally, in the reflexive case, when computing $\inf_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}^{\mathbb{Q}} \left(U \left(\hat{X}_T \right) \right)$ we realize that it is enough to do it over a norm-bounded subset of $d\mathbb{Q}/d\mathbb{P}$. Indeed, we have already proven that $u(x) = \inf_{\mathbb{Q} \in \mathcal{Q}} u_{\mathbb{Q}}(x)$, and this is finite by Assumption 2.3. Thus we may only regard $\mathcal{Q} \cap \{\mathbb{Q} : u_{\mathbb{Q}}(x) \leq u(x) + 1\}$, but by Proposition 5.11 we have that $u_{\mathbb{Q}}(x) \geq c(x) |d\mathbb{Q}/d\mathbb{P}|_I^a$, and so this set is contained in $\mathcal{Q} \cap \{\mathbb{Q} : |d\mathbb{Q}/d\mathbb{P}|_I^a \leq c(x)^{-1} [u(x) + 1]\}$. By reflexivity and Assumption 2.3, these sets are weakly compact (i.e. $\sigma(E_I, L_J)$ -compact) and so the continuous linear functional $Z \mapsto \mathbb{E} \left(Z U \left(\hat{X}_T \right) \right)$ attains its minimum there. Any of these densities along with the optimal \hat{X} conforms a saddle point. We finally stress that the reflexivity condition on L_I is satisfied if the market is complete and Assumption 2.2 holds. Indeed by completeness we would have that $I(\cdot) = \mathbb{E}[\eta_1^*(\cdot)]$ and $J(\cdot) = \mathbb{E}[\eta_1(\cdot)]$, and so by Assumption 2.2 coupled with Proposition 3.11 and Theorem 3.6 we get the desired reflexivity. \square

Remark 5.12. From the previous proof it is clear that if $d\mathbb{Q}/d\mathbb{P} \in E_I$ then at least for the minimax result and the existence of an optimal wealth, the condition $L_I^* \cong L_J$ can be avoided altogether, since we may work with E_I instead of L_I from the beginning, and $E_I^* \cong L_J$ holds.

Let us point out that at the moment we can only prove existence of a worst-case $\hat{\mathbb{Q}}$ (as well as relating it explicitly to the optimal \hat{X}) in the case that our modular spaces are reflexive. In Theorem 5.14 and Remark 5.16, we aim to find out when this is the case. The following property relates the answer to the set \mathcal{Y} .

Lemma 5.13. *If E_J has order-continuous norm (i.e. $|x_\alpha|_J \searrow 0$ whenever $x_\alpha \searrow 0$) then \mathcal{Y} is uniformly integrable.*

Proof. By Theorem 9.22 in [1], E_J has order-continuous norm if and only if every sequence of order-bounded and disjoint elements is strongly convergent to zero. So take A_n a sequence of disjoint sets. Notice that $\mathbb{1}_{A_n}$ is an order-bounded and disjoint sequence, and thus $|\mathbb{1}_{A_n}|_J \rightarrow 0$. This implies $J(\mathbb{1}_{A_n}) \rightarrow 0$, which means $\sup_{Y \in \mathcal{Y}} \mathbb{E}[\mathbb{1}_{A_n} Y] \rightarrow 0$. Now, from Theorem 7 in [16] this implies that \mathcal{Y} is uniformly integrable. \square

The following theorem is essential and it implies Theorem 2.5.

Theorem 5.14. *If the set \mathcal{Y} is not uniformly integrable, then neither E_J , L_J nor E_I can be reflexive.*

Proof. As pointed out in Corollary 9.23 in [1], a reflexive Banach lattice has order continuous norm. Since E_J is a Banach lattice in itself, if it were reflexive, by Lemma 5.13 the set \mathcal{Y} would be uniformly integrable. Thus E_J is not reflexive and therefore L_J neither, since the former is a closed subset of the latter. On the other hand, under the assumption of this section the dual of E_I is isomorphic to L_J (which we proved in Proposition 5.9) which in turn implies that E_I cannot be reflexive either. \square

Remark 5.15. The previous result states that lack of uniform integrability of \mathcal{Y} implies that the space L_I cannot be reflexive. This means that the approach used for Orlicz-Musielak spaces (in the complete case) does not extend vis-à-vis to the current modular space setting. It is remarkable that no growth conditions on U or V may yield reflexivity to our modular spaces as soon as \mathcal{Y} is not uniformly integrable.

Remark 5.16. If the set \mathcal{Y} were uniformly integrable, then also the set of absolutely continuous martingale measure \mathcal{M} would be so (more precisely, their densities would be $\sigma(L^1, L^\infty)$ –relatively compact). Theorem 6.7 and Corollary 7.2 in [13] then say that \mathcal{M} must be a singleton, at least in the case of bounded continuous prices and either if all martingales on the filtration are continuous (e.g. the augmented brownian filtration) or if the filtration is quasi left-continuous. Therefore in most cases uniform integrability of \mathcal{Y} implies completeness.

We envisage that further analysis of our modular spaces (for instance identifying the dual of L_J , or establishing when L_I is a norm-dual space) may bring a better understanding of the robust problem and the (non)existence of the associated worst-case measures. This could be endeavoured through minimization of entropy techniques alternatively.

Appendix

Proof. (Lemma 3.9) The first two items are well-known and can be found in Lemma 2.3.2 in [4]. We prove only the third one here. Clearly $\bar{\gamma}_l(x) = \sup_{z \geq 0} \{|x|z - zV(l/z)\}$. The first order condition for this (assuming $z \neq 0$) is $|x| - V(l/z) + \frac{l}{z}V'(l/z) = 0$. But using that $V' = -[U']^{-1}$ one gets $|x| = U([U']^{-1}(l/z))$ or better $z = \frac{l}{U' \circ U^{-1}(|x|)}$. Therefore $\bar{\gamma}_l(x) = \frac{|x|l}{U' \circ U^{-1}(|x|)} - \frac{l}{U' \circ U^{-1}(|x|)}V \circ U' \circ U^{-1}(|x|)$. Using again the identity $V(y) = U([U']^{-1}(y)) - y[U']^{-1}(y)$ one arrives at $\bar{\gamma}_l(x) = lU^{-1}(|x|)$. By Lemma 3.9 one knows that $\bar{\gamma}_l \geq 0$ and is null only at the origin. Thus if the supremum defining it were attained at 0, since $0V(l/0) = 0$, this shows x must be null. But also $U^{-1}(0) = 0$. Hence, the asserted expression for $\bar{\gamma}_l$ is always valid. \square

Proof. (Lemma 4.2) Let $\widehat{L_{\eta^*}}$ denote the algebraic dual of L_η and $\widehat{L_{\eta^*}}$ its subspace of relatively bounded forms. We extend Φ_y^* to $\widehat{L_{\eta^*}}$ by replacing the expectation in (4.2) by the dual product in $\widehat{L_{\eta^*}} \times L_\eta$ and note that this Φ_y^* corresponds to the function Φ^* in Proposition 5.10 in [28], while space U therein corresponds to space L_η here. Moreover, Φ_+^* and Φ_-^* therein respectively correspond in our setting to $\Phi_{y,+}^*$ and the convex indicator of 0 (since $\gamma(-|\cdot|) = 0$) and part a) of that result we then get $\text{dom } \Phi_y^* = \{\xi \in \widehat{L_{\eta^*}} : \Phi_y^*(\xi) < \infty\} \subseteq \{\xi \in \widehat{L_{\eta^*}} : \xi_- = 0\}$ and $\Phi_y^*(\xi) = \Phi_{y,+}^*(\xi_+) = \Phi_y^*(\xi_+)$ for all $\xi \in \text{dom } \Phi_y^*$.

Notice now on one hand that $L_{\eta^*} \subset \text{dom } \Phi_{y,+}^*$ since $\Phi_{y,+}^*(Z) = \int \gamma_y^*(|Z|)d\mathbb{P} < \infty$ for $Z \in L_{\eta^*}$ and, on the other, $\langle \xi, W/\|W\|_{L_\eta} \rangle \leq \Phi_{y,+}^*(\xi) + y \int \gamma(|W|/\|W\|_{L_\eta})d\mathbb{P} \leq \Phi_{y,+}^*(\xi) + y$ for all $\xi \in \text{dom } \Phi_{y,+}^*$ and $W \in L_\eta \setminus \{0\}$, since $\int \gamma(|W|/\|W\|_{L_\eta})d\mathbb{P} = \int \eta(W/\|W\|_{L_\eta})d\mathbb{P} \leq 1$ (by definition of $\|W\|_{L_\eta}$ and Fatou's Lemma). Taking $-W$ instead of W , we get $|\langle \xi, W \rangle| \leq (\Phi_{y,+}^*(\xi) + y)\|W\|_{L_\eta}$. Thus, we have $\text{dom } \Phi_{y,+}^* = L_{\eta^*}$ and, in the notation of Proposition 5.10 in [28], $L = L_{\eta^*}$, $L_+ = L_{\eta^*}$ and $L_- = \{0\}$. With part b) of that result we get that $\text{dom } \bar{\Phi}_y \subset \widehat{L_\eta}$ and that for all $\zeta \in \text{dom } \bar{\Phi}_y$ the first two equalities in (4.3) hold. Since the Orlicz space L_{η^*} is reflexive, by Theorem 9.11 in [1] we get that $\widehat{L_\eta} = L_\eta$ so that $\text{dom } \bar{\Phi}_y \subset L_\eta$ as claimed. We then easily conclude since $\bar{\Phi}_y$ coincides with Φ_y on L_η . \square

Proof. (Lemma 5.1) We prove (i) first. Call Y^* some element of \mathcal{Y}^* . For any $Y \in \mathcal{Y}$ define $Y^n = \frac{n-1}{n}Y + \frac{1}{n}Y^*$. By convexity $Y^n \in \mathcal{Y}$, and by non-negativity $Y^n \geq \frac{1}{n}Y^*$, implying that $Y^n \in \mathcal{Y}^*$, since V is decreasing. By convexity $\mathbb{E}[|Z|V(Y_T^n/|Z|)] \leq (\frac{n-1}{n})\mathbb{E}[|Z|V(Y_T/|Z|)] + \frac{1}{n}\mathbb{E}[|Z|V(Y_T^*/|Z|)]$, so $\liminf \mathbb{E}[|Z|V(Y_T^n/|Z|)] \leq \mathbb{E}[|Z|V(Y_T/|Z|)]$, and we get that $I(Z) = \inf_{Y \in \mathcal{Y}^*} \mathbb{E}[\eta_Y^*(Z)]$. On the other hand, take $X \in \text{dom}(J)$ and since of course $\frac{n-1}{n}\mathbb{E}[YU^{-1}(X)] + \frac{1}{n}\mathbb{E}[Y^*U^{-1}(X)]$ tends to $\mathbb{E}[YU^{-1}(X)]$, this directly shows that $J(X) = \sup_{Y \in \mathcal{Y}^*} \mathbb{E}[\eta_Y(X)]$. If $J(X) = +\infty$, take $\mathbb{E}[\hat{Y}_m U^{-1}(X)]$ growing to $+\infty$. If these values are finite then the previous argument shows how to approximate them in \mathcal{Y}^* . If (for large enough m) they are infinite, then also $\frac{n-1}{n}\hat{Y}_m + \frac{1}{n}Y^*$ generates an infinite value. Therefore the identity for J always holds.

For condition (ii), one need only observe that $1 \in \mathcal{Y}^*$ and $\mathcal{E}(-\int \lambda dM) \in \mathcal{Y}^*$, respectively. \square

Proof. (Proposition 5.7) The almost-sure order is a partial order. From this both L_I and L_J are ordered vector spaces and lattices, that is, Riesz lattices. Now, because any of the norms defined in this section are lattice norm (i.e. order preserving), both L_I and L_J are Normed Riesz Spaces.

First we prove that both E_I and E_J are closed subspaces of L_I and L_J , in the spirit of the proof of Proposition 3 in [37], Chap. 3.4. Denote F either I or J . We need to show that $\overline{E_F} \subset E_F$. Take $s \in \overline{E_F}$ and $s_n \rightarrow s$ elements in E_F . For a fixed positive k , choose n so that $|s - s_n|_F^l < \frac{1}{2k}$. We then see by convexity and Lemma 5.6 part 1), that

$$F(2k[s - s_n]) = F\left(\frac{2k[s - s_n]|2k[s - s_n]|_F^l}{|2k[s - s_n]|_F^l}\right) \leq |2k[s - s_n]|_F^l \leq 1.$$

Thus, since $ks = \frac{1}{2}(2k[s - s_n]) + \frac{1}{2}[2ks_n]$ we get by convexity that $F(ks) \leq \frac{1}{2}F(2k[s - s_n]) + \frac{1}{2}F(2ks_n) < \infty$. Since this holds for any $k > 0$, we conclude that $s \in E_F$.

Now completeness of E_I and L_J will be proved, showing that both spaces are Banach lattices. For E_I recall (Theorem 9.3 in [1]) that a Normed Riesz space is a Banach Lattice if and only if every positive, increasing Cauchy sequence is norm convergent. Therefore take (Z_n) a positive, increasing Cauchy sequence in E_I (for Luxemburg's norm). By definition (Z_n) converges a.s. to its supremum, which we call Z , and might be ∞ -valued. Since the sequence is Cauchy, there is a $k > 0$ such that $|Z_n|_I^l \leq k$ for every n . By parts (1) and (3) in Lemma 5.6 we have that $\mathbb{E}(Z_n/k) \leq I(Z_n/k) + J(1) \leq 1 + U^{-1}(1)$ implying by Fatou's Lemma that Z is in particular finite, and so Z_n converges to Z in probability (on the non-extended real line). Notice that for every $\lambda > 0$ also $I(\lambda(Z_n - Z_m)) \rightarrow 0$ as (n, m) grows. Indeed, if $\lambda|Z_n - Z_m|_I^l \leq \epsilon < 1$ we have by convexity and Lemma 5.6.(1) that $I(\lambda(Z_n - Z_m)) \leq \lambda|Z_n - Z_m|_I^l \leq \epsilon$. Thus, fixing any $\lambda > 0$ we have for every $\epsilon > 0$ the existence of $N = N(\lambda, \epsilon)$ big enough s.t. $m > n > N$ implies $I(\lambda(Z_m - Z_n)) \leq \epsilon$ and hence taking limit in m by lower-semicontinuity we get $I(\lambda(Z - Z_n)) \leq \epsilon$. Therefore $I(\lambda|Z_n - Z|) \rightarrow 0$ and by part (3) in Lemma 5.6 we see that $Z_n \rightarrow Z$ strongly. By the first part of this proof we finally get that $Z \in E_I$.

Now for L_J , take (X_n) an arbitrary Cauchy sequence. The same sequence is Cauchy in every Orlicz-Musielak space associated to $YU^{-1}(\cdot)$ ($Y \in \mathcal{Y}^*$). Call $\|\cdot\|_Y$ the associated Luxemburg norm. Because these spaces are complete, the sequence norm-converges to (possibly different) limits in each of them. However, since this convergences are stronger than L^0 convergence, the limit must be necessarily (a.s.) unique. Thus, $X_n \rightarrow X$ for every Orlicz-Musielak space associated to η_Y and in probability. By Fatou's lemma $W \mapsto \mathbb{E}[YU^{-1}(W)]$ is lower-semicontinuous in $(L^0)_+$ and thus (as a supremum) also $J(\cdot)$ is so, from which $J(kX) \leq \liminf J(kX_n) \leq 1$ where k^{-1} is an upper bound for the L_J norms of

the (X_n) (it exists because sequence is Cauchy) and by Lemma 5.6.(1). Therefore $X \in L_J$. Evidently $\|X_n - X\|_Y \leq \|X_n - X_m\|_Y + \|X_m - X\|_Y \leq |X_n - X_m|_J^l + \|X_m - X\|_Y$. Now given $\epsilon > 0$ we can make $|X_n - X_m|_J^l \leq \epsilon$ for $n, m \geq N$ independently of $Y \in \mathcal{Y}^*$. On the other hand $\|X_m - X\|_Y \leq \epsilon$ for $m \geq M(Y)$. From here, $\|X_n - X\|_Y \leq 2\epsilon$ for every $n \geq N$ independent of Y . Thus by Lemma 5.6.(1) again, $\mathbb{E}[YU^{-1}(|X_n - X|/[2\epsilon])] \leq 1$ and taking supremum yields $J(|X_n - X|/[2\epsilon]) \leq 1$ also, from which $|X_n - X|_J^l \leq 2\epsilon$ by definition of this norm. Therefore the sequence is convergent.

For the order-continuity of E_I , we need to show that if $Z_\alpha \searrow 0$ a.s. then $|Z_\alpha|_I \searrow 0$. Fix $\beta > 0$ and for a fixed α_0 in the set of indices, notice that $I(\beta Z_{\alpha_0}) < \infty$. Thus there is a Y such that $\mathbb{E}[Z_{\alpha_0}V(Y/(\beta Z_{\alpha_0}))] < \infty$. But $Z_\alpha V(Y/(\beta Z_\alpha))$ decreases to 0 and is dominated by $Z_{\alpha_0}V(Y/(\beta Z_{\alpha_0}))$ (for α big enough, in the sense of the net), which is integrable. By dominated (or monotone) convergence then $\mathbb{E}[Z_\alpha V(Y/(\beta Z_\alpha))] \searrow 0$ and therefore $I(\beta Z_\alpha) \searrow 0$. Since this holds for every $\beta > 0$, by Lemma 5.6.(3) this shows that $|Z_\alpha|_I \searrow 0$. \square

Proof. (Proposition 5.9) Let $l \in (E_I)^*$ and define $\mu(A) := l(\mathbf{1}_A)$ for $A \in \mathcal{F}$ (well-defined and finite by Lemma 5.10.(1)). Clearly $\mu(\emptyset) = 0$. Also if $A_n \in \mathcal{F}$ are disjoint, and writing $A = \cup_n A_n$, then $\sum_{n \leq N} \mathbf{1}_{A_n} \rightarrow \mathbf{1}_A$ a.s. and $|\sum_{n \leq N} \mathbf{1}_{A_n} - \mathbf{1}_A| \leq 1$. Therefore by (3) in Lemma 5.10 then $\sum_{n \leq N} \mathbf{1}_{A_n} \rightarrow \mathbf{1}_A$ in E_I . By continuity of l then $l(\mathbf{1}_A) = \lim_N \sum_{n \leq N} l(\mathbf{1}_{A_n})$. Thus μ is clearly a finite, signed, countably-additive measure. If $A \in \mathcal{F}$ is such that $\mathbb{P}(A) = 0$ then $l(\mathbf{1}_A) = 0$ and hence $\mu(A) = 0$. So μ is absolutely continuous w.r.t. \mathbb{P} . By Radon-Nikodym's Theorem then $g := \frac{d\mu}{d\mathbb{P}}$ exists and is \mathbb{P} -integrable. By linearity then $l(f) = \mathbb{E}[fg]$ for every simple function f . By continuity $|\mathbb{E}[fg]| \leq C|f|_I$ for simple functions. Therefore $\sup\{|\mathbb{E}[fg]| : f \text{ simple and } |f|_I^q \leq 1\} < \infty$ and by (4) in Lemma 5.10 we get that $g \in L_J$ and that $|g|_J^l$ equals the above supremum. Since both $l(\cdot)$ and $\mathbb{E}(\cdot g)$ are uniformly continuous functions coinciding on a dense set (by (2) in Lemma 5.10, simple functions are such a set), they must agree in the whole of E_I . Hence $l(f) = \mathbb{E}[fg]$ for every $f \in E_I$ and so $(E_I)^* \subset L_J$, but using Proposition 5.8 the reverse inclusion already holds. Therefore $(E_I)^* = L_J$, where the identification is isomorphic if L_J is endowed with the Luxemburg norm and E_I with the Amemiya one. Now take $X \in L_J$ and call $l_X(\cdot) := \mathbb{E}[X\cdot]$. Then:

$$\begin{aligned} I^*(l_X) &= \sup_{Z \in L_I} \left\{ \mathbb{E}[XZ] - \inf_{Y \in \mathcal{Y}^*} \mathbb{E} \left[|Z|V \left(\frac{Y}{|Z|} \right) \right] \right\} = \sup_{Y \in \mathcal{Y}^*} \sup_{Z \in L_I} \left\{ \mathbb{E}[XZ] - \mathbb{E} \left[|Z|V \left(\frac{Y}{|Z|} \right) \right] \right\} \\ &= \sup_{Y \in \mathcal{Y}^*} \sup_{Z \in L_{\eta_Y^*}} \left\{ \mathbb{E}[XZ] - \mathbb{E} \left[|Z|V \left(\frac{Y}{|Z|} \right) \right] \right\} = \sup_{Y \in \mathcal{Y}^*} \mathbb{E}[YU^{-1}(X)] \\ &= J(X) \end{aligned}$$

since the conjugate of η_Y^* is η_Y . Now fix $Z \in L_I$ and assume $L_I = E_I$. Then $J^*(l_Z) = \sup_{X \in L_J} \{\mathbb{E}[XZ] - I^*[l_X]\}$ by the previous lines. On the other hand, $I(Z) = \sup_{l \in (L_I)^*} \{l(Z) - I^*(l)\} = \sup_{X \in L_J} \{\mathbb{E}[XZ] - I^*[l_X]\}$, since $(E_I)^* = L_J$. Thus $J^*(l_Z) = I(Z)$. \square

Proof. (Lemma 5.10) For the first point, $\mathbf{1}_A \in E_I$ iff $\inf_{Y \in \mathcal{Y}} \mathbb{E}[\mathbf{1}_A V(\beta Y)] < \infty$ for every $\beta > 0$. This is true, simply by taking a $Y \in \mathcal{Y}^*$.

For the third point, if $|Z_n| \leq K$, then $I(\alpha Z_n) \leq \alpha \inf_{Y \in \mathcal{Y}} \mathbb{E}[KV(Y/(\alpha K))]$. But we have $|Z_n|V(Y/\alpha|Z_n|) \rightarrow 0$ a.s. and this sequence is dominated by $KV(Y/(\alpha K))$. Therefore if there exists a $Y \in \mathcal{Y}$ such that $\mathbb{E}[V(Y/(\alpha K))] < \infty$, then it would follow that $I(\alpha Z_n) \rightarrow 0$. But this holds (for every $\alpha > 0$) again by taking $Y \in \mathcal{Y}^*$. By Lemma 5.6.(3) we conclude that $Z_n \rightarrow 0$ strongly.

The proof of the second point resembles the previous one. First, since simple functions are dense in L^∞ and by Proposition 5.8 this last space is contained continuously in L_I (obviously then also in E_I), it suffices to show that bounded functions are dense in E_I . Take $Z \in E_I$ and define $Z_n = Z\mathbf{1}_{|Z| < n}$. Thus $X_n := |Z - Z_n| = |Z|\mathbf{1}_{|Z| \geq n} \searrow 0$ a.s. Now fix $\beta > 0$. Taking any $N > 0$ and because $\infty > I(\beta X_N) = \beta \mathbb{E}[X_N V(Y/(\beta X_N))]$ for some $Y \in \mathcal{Y}$, and $X_n V(Y/(\beta X_n)) \searrow 0$ a.s. then by dominated (or monotone) convergence $\mathbb{E}[X_n V(Y/(\beta X_n))] \rightarrow 0$ and thus $I(\beta X_n) \rightarrow 0$. Now because this holds for every β , by Lemma 5.6.(3) then $|X_n|_I \rightarrow 0$.

Finally, for the fourth point, take $\kappa < \infty$ as in the statement. Then clearly $\sup\{|\mathbb{E}[zg]| : z \text{ simple and } \|z\|_{\eta_Y^*}^a \leq 1\} \leq \kappa$ for every $Y \in \mathcal{Y}^*$. A classical result in Orlicz theory (see (10) in Proposition 10, [37], chapter 3.4), which readily generalizes to Orlicz-Musielak spaces, implies that $\|g\|_{\eta_Y}^l = \sup\{|\mathbb{E}[zg]| : \|z\|_{\eta_Y^*}^a \leq 1\}$, and hence $\|g\|_{\eta_Y}^l \leq \kappa$, since any non-negative z may be approximated in an increasing way a.s. by simple functions. Hence $\sup_{Y \in \mathcal{Y}^*} \|g\|_{\eta_Y}^l \leq \kappa$. Since $\mathbb{E}[YU^{-1}(g/\|g\|_{\eta_Y}^l)] \leq 1$ (by definition of the norm and Fatou's Lemma) then $\mathbb{E}[YU^{-1}(g/\kappa)] \leq 1$ and thus $J(g/\kappa) \leq 1$ from which $|g|_J^l \leq \kappa < \infty$. Finally, by Proposition 5.8 we have $|\mathbb{E}[fg]| \leq |g|_J^l |f|_I^a$ and so if f is simple and such that $|f|_I^a \leq 1$ we get that $|\mathbb{E}[fg]| \leq |g|_J^l$ and then by taking supremum over such functions we derive that $\kappa \leq |g|_J^l$, and therefore there is equality. \square

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